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Darboux First Integral Conditions and Integrability of the 3D Lotka-Volterra System

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Abstract

We apply the Darboux theory of integrability to polynomial ODE's of dimension 3. Using this theory and computer algebra, we study the existence of first integrals for the 3–dimensional Lotka–Volterra systems with polynomial invariant algebraic solutions linear and quadratic and determine numerous cases of integrability.

1 Introduction

In this paper we are concerned with the 3-*dimensional Lotka-Volterra [1,2] system (LV3)*, having first integrals formed with linear and quadratic polynomials. The LV3 is

$$\begin{aligned}
\dot{x} &= x(a_1 + b_{11}x + b_{12}y + b_{13}z) = P_1(x,y,z),\\
\dot{y} &= y(a_2 + b_{21}x + b_{22}y + b_{23}z) = P_2(x,y,z),\\
\dot{z} &= z(a_3 + b_{31}x + b_{32}y + b_{33}z) = P_3(x,y,z),
\end{aligned} \tag{1.1}$$

where the overdot means time derivative. This system typically model the time evolution of conflicting species in chemistry and biology [3], where the linear terms a_i (also called Malthusian terms) denote growth (or decay) rates of each species independently of the others; the self-interactive terms, also called Verhulst terms (b_{ii}) represent the control on over-population of each of the respective species (such as cannibalism or depletion of resources), and the cross-interactive terms ($b_{ij}, i \neq j$) represent inter-species interactions (such as predator-prey). It has been extensively studied, starting with the pioneer works of Lotka [1] and Volterra [2], in the case where all the Verhulst terms vanish. Then in three dimensions the essential parameters reduce to only 6. A detailed study of this case can be found in Ref. 4

There are many other natural phenomena modeled by (1.1), such as the coupling of waves in laser physics [5], the evolution of electrons, ions and neutral species in plasma physics [6]. In hydrodynamics, they model the convective instability in the Bénard problem [7]. Similarly, they appear in the interaction of gases in a background host medium [8].

In the theory of partial differential equations they can be obtained as a discretized form of the Korteweg-de Vries equation [9]. They also play a role in such diverse topics of current interest as neural networks [10], biochemical reactions etc. The systems interest became crucial after the work of Brenig and Goriely [11,12] wherein they prove that a large class of ordinary differential equations implied in various fields of physics, biology, chemistry and economics, can be transformed into a Lotka-Volterra of greater order using a quasimonomial formalism. In the context of plasma physics, all the nonlinear terms represent binary interactions or model certain transport across the boundary of the system.

In solving for the LV3, it is worthwhile to know, given a set of initial conditions, what its long-time asymptotic behaviour will be or whether stable periodic solutions exist. The existence of stable periodic orbits would be rather important for experimentalists wishing to obtain and maintain a stable oscillatory state. Since in general the solutions of the LV3 cannot be written in terms of elementary functions, the two questions of asymptotic behaviour and the existence of periodic orbits are rather hard to answer. Nor it is easy to explore the general solution using numerical schemes since one has to prescribe all the parameters of the LV3 in terms of real numbers. It will then be of most interest to possess *constants of the motion*, which, as such, contain the trajectories, and permits the localisation of the solutions or find the asymptotic state. Particularly, if a three-dimensional dynamical system admits a constant of motion, then the phase space is foliated into two-dimensional leaves and therefore certain types of irregular orbits cannot occur. In some cases it is even possible to find for a given three-dimensional dynamical system two functionally independent constants of the motion. Then the orbits are the intersections of two-dimensional flow invariant solutions and therefore nonchaotic. (Indeed, chaotic behaviour is associated with nonintegrability of the dynamical system.) In fact, obtaining a constant of motion corresponds to a partial integration and is interesting both from an analytical and numerical point of view. In this last point of view, obtaining a constant of motion is equivalent to reducing by one unit the dimension of the phase space. Since in nonlinear problems we are usually interested in a full exploration of initial conditions, any reduction of the dimension corresponds to a dramatic saving of numerical computation. Moreover, the existence of a constant of motion provides a welcome check of the numerical scheme with respect to its accuracy and stability.

To introduce the adopted terminology, we must note that a constant of motion may be time-dependent. Usually this time-dependency appears in an exponential form [13,14]. Some authors [14] use the name of *invariants*, name which they apply regardless of the type of constant of motion. As here we are concerned only with time-independent constants of motion, we will keep for them the more usual name of *first integrals*. Much research effort has been devoted to the problem of finding time-independent and time-dependent constants of motion for hamiltonian and nonhamiltonian ordinary differential equations. The reason is that except for some simple cases, this problem is very hard and no general methods to solve it are known up to now. Nevertheless several approaches were developed in the last years, thanks to the use of computer algebra facilities. The most important among them being: specific ansatz for a invariant [13-16], singularity analysis [17,18], the Lie symmetry method [19,20], the linear compatibility analysis method [21,22], rescaling method [23,24] and the Darboux method [25-28]. Among the first mentioned methods one can cite the Carleman embedding procedure [13], the generalised Carleman [14] and the Hamiltonian methods [15,16]. Although they differ in the details of computation, all

these methods are based on an a priori hypothesis on the form of the invariant or the first integral H.

In 1878 Darboux [25] showed how can be constructed the first integrals of planar polynomial ordinary differential equations possessing sufficient invariant algebraic curves. The Darboux method is based on the possibility of writing the invariant (or at least an integrating factor) as the product of different algebraic functions f_i raised at a given power λ_i. It is on the form of these functions f_i that it is introduced an Ansatz. The f_i are determined straightforwardly identifying to zero the coefficients of a polynomial expression in x, y, z . In that sense the Darboux method is not so different from some of the others cited above. Nevertheless the experiance shows that we have somehow divided the difficulties of the unavoidable identification. Typically the number of equations obtained is greater than the number of unknowns. In order to satisfy the whole system of algebraic equations we must then introduce some *conditions* on the parameters describing the system (1.1). From a physical point of view the rule of the game is to find first integrals with as few conditions as possible on the parameters of the given system.

Although the Darboux theory of integrability works for real or complex polynomial ordinary differential equations, we are concerned here with the existence of real first integrals of (1.1) when the parameters a_i, b_{ij} of the system, the three dependent variables x, y, z, and the independent variable t (the *time*) are real.

The paper is organized as follows. The main lines of the Darboux theory for three-dimensional polynomial differential systems are presented in Section 2. The first integrals constructed using the Darboux theory are build here exclusively with polynomial invariant algebraic solutions. In Section 3, the Proposition 1 exhibits the linear algebraic solutions of the LV3. The following sections contain the cases for which at least a first integral has been found. The cases where the LV3 has a single first integral are considered first, see Sections 4 and 5. In Section 4, the Theorems 2 and 3 concern the cases where all the f_i are linear and in Section 5, the Theorems 4 and 5 are for the cases where f_4 is a quadratic polynomial, the first being when the conic passes through the origin and the second when not. In Section 6 we consider the cases of integrability, i.e. the cases where two first integrals coexist. Theorems 6–10 are when at most one of the first integrals contain quadratic algebraic solutions and Theorem 11 is for the cases where the two first integrals are formed with a quadratic algebraic solution. Finally in Section 7 we give our conclusions.

2 Computational method

The Darboux method is based on the existence of algebraic invariant solutions. Suppose that we can determine two polynomials $f_i(x, y, z)$ and $K_i(x, y, z)$ such that

$$\frac{\partial f_i}{\partial x} P_1 + \frac{\partial f_i}{\partial y} P_2 + \frac{\partial f_i}{\partial z} P_3 = K_i f_i \ . \tag{2.1}$$

Then equation $f_i(x, y, z) = 0$ describes a surface which is formed by trajectories. It is an *algebraic solution* of the system. The polynomial K_i is called the *cofactor* of f_i.

These f_i are going to be the "bricks" with which we will build the invariants. Suppose that we have obtained q functions f_i. Let us consider the following function of the variables

x, y, z and t

$$H = \prod_{i=1}^{q} f_i^{\lambda_i}(x, y, z)\exp(st) \ . \tag{2.2}$$

Taking the total derivative

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}P_1 + \frac{\partial H}{\partial y}P_2 + \frac{\partial H}{\partial z}P_3 \\ &= H\left[s + \sum_{i=1}^{q}\frac{\lambda_i}{f_i}\left(\frac{\partial f_i}{\partial x}P_1 + \frac{\partial f_i}{\partial y}P_2 + \frac{\partial f_i}{\partial z}P_3\right)\right] \ . \end{aligned} \tag{2.3}$$

Imposing that H is an invariant and making use of (2.1), we obtain

$$s + \sum_{i=1}^{q}\lambda_i K_i = 0 \ . \tag{2.4}$$

The equations in the λ_i are now linear equations. How many f_i do we need? The equation (2.1) shows that if the system is of degree m, K_i is at most of degree $m-1$ independently of the degree of f_i. So, the left hand side of (2.4) is a polynomial in x, y, z of degree at most $m-1$ with a total of $(m+2)(m+1)m/6$ terms. Consequently (2.4) produces $(m+2)(m+1)m/6$ equations where the unknowns are the λ_i. We see from (2.4) that if we allow s to be different from zero the system of equations in the λ_i is inhomogeneous, and we need $q = (m+2)(m+1)m/6$. If we want a first integral, then $s = 0$ and the system becomes homogeneous and we need a priori either a new f_i, or a new condition by imposing that the determinant of the system of equations in the λ_i is zero. This will be the case here where we are concerned with first integrals exclusively. The required number of λ_i in order to have first integrals is then $q = (m+2)(m+1)m/6 + 1$. However, if it exists a solution with $q = (m+2)(m+1)m/6$, then a first integral exists. In fact, all the first integrals presented here are with this value of q (Theorem 3 is for first integrals having a quadratic algebraic solutions which factorize to linear ones). We see that the possibility of solving for the λ_i in the system deduced identifying to zero the coefficients of the polynomial on x, y, z deduced from equation (2.4) depends on the conditions we put on the coefficients of (1.1). We should not forget that some conditions come from equation (2.4) and others from equation (2.1).

We make a comment on the number of conditions on the system parameters in order to obtain a first integral and its variation with the degree of the invariant solutions tested. For the existence of invariant solutions of higher degree we require a larger number of conditions than for solutions of smaller degree. At least this is what happens for degrees one to three. For higher degrees one can get a saturation as this is what happens in 2d (see Ref 29). It turns out that the most interesting cases (i.e. the cases with a not too high number of conditions) will be obtained using straight lines and conics as invariant solutions for Lotka–Volterra and quadratic systems.

In the case of system (1.1), $m = 2$ and in this paper we take $q = 4$. As the axes are algebraic solutions, the problem of invariant search is reduced to the determination of only one algebraic solution i.e. $f_4 = 0$. Moreover f_4 is taken here as polynomial of degree at most two, namely $f_4 \equiv (\overrightarrow{\nu} \cdot \overrightarrow{f})$ where $\overrightarrow{\nu}$ is a vector of dimension 10 and with components 1 or 0

and $\overrightarrow{f}$ a vector of components $(f_{000}, f_{100}x, f_{010}y, f_{001}z, f_{200}x^2, f_{110}xy, f_{101}xz, f_{020}y^2, f_{011}yz, f_{002}z^2)$.

The results presented here are given modulo the *three-dimensional Lotka-Volterra system equivalences*. This is because we can associate to a given LV3 (1.1) five equivalent three-dimensional Lotka-Volterra systems. The first two are obtained doing circular permutation of the variables x, y, z and of the parameters a_i and b_{ij}, next three systems $\forall b_{ij} \neq 0$ are obtained doing the transformation

$$(x,\ y,\ z,\ a_1,\ a_2,\ a_3,\ b_{11},\ b_{12},\ b_{13},\ b_{21},\ b_{22},\ b_{23},\ b_{31},\ b_{32},\ b_{33}) \to$$
$$(x,\ z,\ y,\ a_1,\ a_3,\ a_2,\ b_{11},\ b_{13},\ b_{12},\ b_{31},\ b_{33},\ b_{32},\ b_{21},\ b_{23},\ b_{22}),$$

which keeps invariant the system, and the two others obtained doing circular permutation of variables and parameters. We say that all these Lotka–Volterra systems are *E equivalent*. All the results of this paper are stated modulo these *E equivalences*.

Table 1. Definition of the terminology used for the first integral conditions.

na.	definition	na.	definition
a_i	a_i	i_{ij}	$b_{ik} - b_{jk} - b_{kk}$
c_{ij}	$a_i - a_j$	i'_{ij}	$b_{ik} + b_{jk} - b_{kk}$
c'_{ij}	$a_i - 2a_j$	i''_{ij}	$2b_{ik} - b_{jk} - 2b_{kk}$
c''_{ij}	$a_i + a_j$	i'''_{ij}	$2b_{ik} - b_{jk} - b_{kk}$
c'''_{ij}	$a_i + 2a_j$	j_{ij}	$b_{ik} + b_{jk} - 2b_{kk}$
v_{ijk}	$a_i + a_j + a_k$	j'_{ij}	$b_{ik} + b_{jk} - 4b_{kk}$
v'_{ijk}	$a_i + a_j - a_k$	j''_{ij}	$2b_{ik} + b_{jk} - 4b_{kk}$
v''_{ijk}	$a_i + a_j - 2a_k$	n_{ij}	$b_{ii}(b_{ij} - b_{jj}) - (b_{ij} - 2b_{jj})(b_{ji} - b_{ii})$
v'''_{ijk}	$a_i + a_j + 2a_k$	n'_{ij}	$(b_{ij} - b_{jj})(b_{ki} - b_{ii}) + (b_{ij} + b_{kj} - 2b_{jj})(b_{ji} - b_{ii})$
w_{ijk}	$2a_i + a_j + a_k$	n''_{ij}	$b_{ii}(b_{ij} + b_{kj} - 3b_{jj}) - b_{ki}(b_{ij} - b_{jj})$
w'_{ijk}	$2a_i + a_j - a_k$	n'''_{ij}	$3b_{ii}b_{jj} + b_{ji}b_{kj} - b_{ii}(b_{ij} + b_{kj}) - b_{jj}(b_{ji} + b_{ki})$
w''_{ijk}	$2a_i - a_j - a_k$	n^{iv}_{ij}	$(b_{ij} - b_{jj})(b_{ki} - 2b_{ii}) - b_{jj}(b_{ji} - b_{ii})$
w'''_{ijk}	$a_i + 2(a_j + a_k)$	o_{ijk}	$(b_{ij} - b_{jj})(b_{jk} - b_{kk})(b_{ki} - b_{ii}) + (b_{ji} - b_{ii})(b_{kj} - b_{jj})(b_{ik} - b_{kk})$
b_i	b_i	p_{ij}	$2b_{ii}(b_{kj} - b_{jj}) + b_{jj}(b_{ji} - b_{ii})$
d_{ij}	$b_{ij} - b_{jj}$	p'_{ij}	$2b_{ii}b_{jj} - b_{ij}(b_{ki} + b_{ii})$
d'_{ij}	$2b_{ij} - b_{jj}$	p''_{ij}	$b_{ki}b_{kj} - 2b_{ij}(2b_{ki} - b_{ji})$
d''_{ij}	$b_{ij} - 2b_{jj}$	p'''_{ij}	$b_{ii}b_{kj} - b_{ki}(b_{kj} - b_{jj})$
e_{ij}	$b_{ij} + b_{jj}$	p^{iv}_{ij}	$2b_{ii}(b_{ij} - b_{jj}) + b_{jj}(b_{ki} - b_{ii})$
e'_{ij}	$2b_{ij} + b_{jj}$	q_{ij}	$b_{ij}b_{ii}b_{kk} + b_{ji}b_{jj}(b_{ik} - 2b_{kk})$
e''_{ij}	$b_{ij} + 2b_{jj}$	q'_{ij}	$b_{ii}b_{ik}(b_{ij} - b_{jj}) + b_{jj}b_{jk}(b_{ji} - b_{ii})$
g_{ij}	$b_{ik} - b_{jk}$	q''_{ij}	$b_{ii}b_{jj}(b_{kk} + b_{jk}) + 2b_{ki}(b_{ij} - 2b_{jj})$
g'_{ij}	$b_{ik} - 2b_{jk}$	q'''_{ij}	$b_{ii}b_{jj}(b_{ki} - 2b_{ii}) - (b_{kj} - 2b_{jj})(b_{ji} - 2b_{ii})$
g''_{ij}	$b_{ik} + b_{jk}$	r_{ij}	$a_ib_{jj}(b_{ji} - b_{ii}) + a_jb_{ii}(b_{ij} - b_{jj})$
g'''_{ij}	$b_{ik} + 2b_{jk}$	r'_{ij}	$a_jb_{ij} - (a_i + 2a_j)b_{jj}$
h_{ij}	$b_{ii}b_{kj} - b_{jj}b_{ki}$	r''_{ij}	$a_jb_{ij} - (a_i + a_j)b_{jj}$
h'_{ij}	$b_{ii}b_{kj} - b_{jj}b_{ji}$	r'''_{ij}	$2a_ib_{ji} - (a_i + a_j)b_{ii}$
k_{ij}	$a_jb_{ij} - a_ib_{jj}$	r^{iv}_{ij}	$a_ib_{jk} - a_j(b_{ik} + 2b_{jk})$
k'_{ij}	$a_jb_{ij} + a_ib_{jj}$	s_{ij}	$a_jb_{ij} - (a_i + a_j + a_k)b_{jj}$
l_{ij}	$a_ib_{jk} - a_jb_{ik}$	s'_{ij}	$2a_jb_{ij} - (a_i + 2a_j + a_k)b_{jj}$
l'_{ij}	$a_ib_{jk} + a_jb_{ik}$	s''_{ij}	$(a_i + 2a_j)b_{jj} - a_j(b_{ij} + b_{kj})$
m_{ij}	$a_jb_{ij} + a_kb_{jj}$	s'''_{ij}	$(a_ib_{ji} - a_jb_{ii})(b_{kj} - b_{jj})(b_{jk} - b_{kk}) + a_ib_{ii}(b_{jk}b_{kj} - b_{jj}b_{kk})$

3 Invariant algebraic solutions

In this section we study the invariant algebraic solutions of the LV3 systems of degree at most 1. Thus, in Proposition 1, we present the invariant planes (invariant algebraic solutions of degree 1) together with their cofactors, and the conditions for their existence.

Proposition 1. *An LV3 has an invariant plane $f = 0$ with cofactor K in the following cases, modulo the E equivalences, using the notation defined in Table 1*

(1) $f = x = 0$ *with* $K = b_{11}x + b_{12}y + b_{13}z$.

(2) *If* $b_{12} = b_{13} = 0$ *and* $a_1 b_{11} \neq 0$, *then* $f = a_1 + b_{11}x = 0$ *and* $K = b_{11}x$.

(3) *If* $a_1 = a_2$, $b_{13} = b_{23}$ *and* $d_{12}d_{21} \neq 0$ *then* $f = (b_{21} - b_{11})x - (b_{12} - b_{22})y = 0$ *and* $K = a_1 + b_{11}x + b_{22}y + b_{13}z$.

(4) *If* $a_1 = a_2 = a_3$, $o_{123} = 0$ *and* $d_{12}d_{21}d_{13}d_{31} \neq 0$, *then* $f = (b_{21} - b_{11})(b_{31} - b_{11})x - (b_{12} - b_{22})(b_{31} - b_{11})y - (b_{13} - b_{33})(b_{21} - b_{11})z = 0$ *and* $K = a_1 + b_{11}x + b_{22}y + b_{33}z$.

(5) *If* $r_{12} = r_{23} = r_{31} = 0$ *and* $a_1 a_2 a_3 b_{11} b_{22} b_{33} \neq 0$, *then* $f = a_1 a_2 a_3 + a_2 a_3 b_{11} x + a_1 a_3 b_{22} y + a_1 a_2 b_{33} z = 0$ *and* $K = b_{11}x + b_{22}y + b_{33}z$.

(6) *If* $b_{13} = b_{23} = r_{12} = 0$ *and* $a_1 a_2 b_{11} b_{22} \neq 0$, *then* $f = a_1 a_2 + a_2 b_{11} x + a_1 b_{22} y = 0$ *and* $K = b_{11}x + b_{22}y$.

(7) *If* $a_3 = b_{33} = r_{12} = o_{123} = 0$ *and* $a_1 a_2 b_{11} b_{22} \neq 0$, *then* $f = (b_{31} - b_{11})(a_1 a_2 + a_2 b_{11} x - a_1 b_{22} y) - a_2 b_{11} b_{13} z = 0$ *and* $K = b_{11}x + b_{22}y$.

(8) *If* $a_2 = b_{22} = b_{13} = b_{23} = 0$ *and* $a_1 b_{11} b_{12} d_{21} \neq 0$, *then* $f = (b_{21} - b_{11})(a_1 + b_{11}x) - b_{11} b_{12} y = 0$ *and* $K = b_{11}x$.

(9) *If* $a_2 = a_3 = b_{22} = b_{33} = o_{123} = 0$ *and* $a_1 b_{11} b_{32} d_{21} \neq 0$, *then* $f = b_{32}(b_{21} - b_{11})(a_1 + b_{11}x) - b_{11} b_{12}(b_{32}y - b_{23}z) = 0$ *and* $K = b_{11}x$.

(10) *If* $a_2 = a_3 = b_{22} = b_{33} = b_{23} = b_{32} = 0$ *and* $a_1 b_{11} d_{21} d_{31} \neq 0$, *then* $f = (b_{21} - b_{11})(b_{31} - b_{11})(a_1 + b_{11}x) + b_{11}[b_{12}(b_{11} - b_{31})y + b_{13}(b_{11} - b_{21})z] = 0$ *and* $K = b_{11}x$.

(11) *If* $a_2 = a_3 = b_{22} = b_{33} = b_{12} = b_{32} = 0$ *and* $a_1 b_{11} d_{31} \neq 0$, *then* $f = (b_{31} - b_{11})(a_1 + b_{11}x) - b_{11} b_{13} z = 0$ *and* $K = b_{11}x$.

(12) *If* $a_2 = a_3 = b_{22} = b_{33} = b_{12} = d_{21} = 0$ *and* $a_1 b_{11} b_{23} b_{32} d_{31} \neq 0$, *then* $f = b_{23}(b_{31} - b_{11})(a_1 + b_{11}x) + b_{11} b_{13}(b_{32}y - b_{23}z) = 0$ *and* $K = b_{11}x$.

Proof. The proof is obtained finding both the linear f and K satisfying equation (2.1). So doing, we introduce in (2.1) the general form for f and K, namely $f = f_{000} + f_{100}x + f_{010}y + f_{001}z$, $K = k_{000} + k_{100}x + k_{010}y + k_{001}z$. The problem consits then in the evaluation of the coefficients of f and K by solving the algebraic system obtained identifying to zero the coefficients of the polynomial in x, y, z which results after this introduction of f and K in (2.1). The number of equations of this algebraic system is usely greater than the number of the unknown coefficients of f and K. In order to satisfy the entire system of algebraic equations, it is necessary then to introduce the conditions on the parameters of the differential system (1.1), which appear in the different statements of the theorem. ∎

4 First integrals formed with linear invariant algebraic solutions

In this section we apply the Darboux method to deduce first integrals obtained with linear polynomial algebraic solutions.

Theorem 2. *Let $f_1 = x$, $f_2 = y$ and $f_3 = z$. If a LV3 has a fourth algebraic solution $f_4 = 0$ of degree 1, then using the notation defined in Table 1, we can establish the following statements*

(1) *If the LV3 satisfy to the conditions* $\det(b_{ij}) = 0$ *and $s = \sum_{i=1}^{3} a_i\alpha_i = 0$, where the α_i are any non-trivial solution of $(b_{ij})^T(\alpha_i) = 0$, then the function $|x|^{\alpha_1}|y|^{\alpha_2}|z|^{\alpha_3}$ is a first integral.*

(2) *If the conditions $c_{12} = c_{23} = o_{123} = 0$ are satisfied and $d_{12}d_{21}d_{13}d_{31} \neq 0$, then the plane $f_4 \equiv \phi = (b_{21} - b_{11})(b_{31} - b_{11})x - (b_{12} - b_{22})(b_{31} - b_{11})y - (b_{13} - b_{33})(b_{21} - b_{11})z = 0$ is an algebraic solution of the LV3 and a first integral $|f_1|^{\alpha_1}|f_2|^{\alpha_2}|f_3|^{\alpha_3}f_4$ exists, the α_i being defined as solution of the system $(b_{ij})^T(\alpha_i) = -\,diag(b_{ij})$.*

(3) *If the conditions $c_{12} = g_{12} = k_{13} = 0$ are satisfied and $d_{12}d_{21}(|b_{11}| + |b_{21}| + |b_{31}|) \neq 0$, then the plane $f_4 = (b_{21} - b_{11})x + (b_{22} - b_{12})y = 0$, is an algebraic solution of the LV3 and a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3}|f_4|^{\lambda_4}$ exists with $\lambda_1 = (b_{21} - b_{11})(b_{13}b_{32} - b_{33}b_{22}), \lambda_2 = (b_{12} - b_{22})(b_{13}b_{31} - b_{33}b_{11}), \lambda_3 = b_{13}(b_{12} - b_{22})(b_{11} - b_{21}), \lambda_4 = b_{13}[b_{32}(b_{11} - b_{21}) + b_{31}(b_{22} - b_{12})] + b_{33}(b_{12}b_{21} - b_{11}b_{22})$.*

(4) *If the conditions $b_{13} = b_{23} = r_{12} = 0$ are satisfied and $a_1a_2b_{11}b_{22}d_{12}d_{21} \neq 0$, then, the plane $f_4 = a_1a_2 + a_2b_{11}x + a_1b_{22}y = 0$, is an algebraic solution of the LV3 and a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}f_3^{\lambda_3}|f_4|^{\lambda_4}$ exists with $\lambda_1 = b_{22}(b_{21} - b_{11}), \lambda_2 = b_{11}(b_{12} - b_{22}), \lambda_3 = 0, \lambda_4 = b_{11}b_{22} - b_{12}b_{21}$.*

(5) *If the conditions $r_{12} = r_{23} = r_{31} = a_1a_3b_{11}b_{22}d_{23} + a_1a_2b_{22}b_{33}d_{31} + a_2a_3b_{11}b_{33}d_{12} = 0$ are satisfied and $a_1a_2a_3b_{11}b_{22}b_{33} \neq 0$, then the plane $f_4 = a_1a_2a_3 + a_2a_3b_{11}x + a_1a_3b_{22}y + a_1a_2b_{33}z = 0$ is an algebraic solution of the LV3 and a first integral $|f_1|^{\alpha_1}|f_2|^{\alpha_2}|f_3|^{\alpha_3}f_4$ exists, the α_i being defined as solution of the system $(b_{ij})^T(\alpha_i) = -\,diag(b_{ij})$.*

(6) *If the conditions $b_{13} = b_{23} = d_{12} = d_{21} = 0$ are satisfied and $a_1a_2c_{12} \neq 0$, then, the plane $f_4 = a_1a_2 + a_2b_{11}x + a_1b_{22}y = 0$, is an algebraic solution of the LV3 and a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}f_3^{\lambda_3}|f_4|^{\lambda_4}$ exists with $\lambda_1 = a_2, \lambda_2 = -a_1, \lambda_3 = 0, \lambda_4 = a_1 - a_2$.*

(7) *If the conditions $a_2 = b_{13} = b_{22} = b_{23} = 0$ and $b_{11}b_{12}b_{21}d_{21} \neq 0$, then the plane $f_4 = (b_{21} - b_{11})(a_1 + b_{11}x) - b_{11}b_{12}y = 0$, is an algebraic solution of the LV3 and a first integral $f_1^{\lambda_1}|f_2|^{\lambda_2}f_3^{\lambda_3}|f_4|^{\lambda_4}$ exists with $\lambda_1 = 0, \lambda_2 = -b_{11}, \lambda_3 = 0, \lambda_4 = b_{21}$.*

(8) *If the conditions $b_{12} = b_{13} = b_{22} = b_{23} = 0$ are satisfied and $a_1b_{11}k_{21} \neq 0$, then the plane $f_4 = a_1 + b_{11}x = 0$, is an algebraic solution of the LV3 and a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}f_3^{\lambda_3}|f_4|^{\lambda_4}$ exists with $\lambda_1 = a_2b_{11}, \lambda_2 = -a_1b_{11}, \lambda_3 = 0, \lambda_4 = a_1b_{21} - a_2b_{11}$.*

(9) *If the conditions $c'_{32} = b_{12} = b_{13} = d'_{23} = d''_{32} = i'''_{23} = 0$ are satisfied and $a_1b_{11} \neq 0$, then the plane $f_4 = a_1 + b_{11}x = 0$, is an algebraic solution of the LV3 and a first integral $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}f_4^{\lambda_4}$ exists with $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 1, \lambda_4 = 1$.*

(10) *If the conditions $v'_{231} = b_{12} = b_{13} = e_{23} = e_{32} = j_{23} = 0$ are satisfied and $a_1b_{11} \neq 0$, then the plane $f_4 = a_1 + b_{11}x = 0$, is an algebraic solution of the LV3 and a first integral $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}f_4^{\lambda_4}$ exists with $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 1$.*

(11) *If the conditions* $c''_{23} = b_{12} = b_{13} = e_{23} = e_{32} = j_{23} = 0$ *are satisfied and* $a_1 b_{11} \neq 0$, *then the plane* $f_4 = a_1 + b_{11}x = 0$, *is an algebraic solution of the LV3 and a first integral* $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ *exists with* $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 2$.

(12) *If the conditions* $c_{23} = b_{12} = b_{13} = b_{23} = b_{33} = d_{31} = d_{32} = 0$ *are satisfied and* $a_1 b_{11} d_{21} \neq 0$, *then the plane* $f_4 = a_1 + b_{11}x = 0$, *is an algebraic solution of the LV3 and a first integral* $f_1^{\lambda_1} |f_2|^{\lambda_2} |f_3|^{\lambda_3} |f_4|^{\lambda_4}$ *exists with* $\lambda_1 = 0, \lambda_2 = b_{11}, \lambda_3 = -b_{11}, \lambda_4 = b_{11} - b_{21}$.

(13) *If the conditions* $c_{31} = c'_{21} = b_{12} = b_{13} = d_{21} = d_{31} = d'_{32} = d''_{23} = 0$ *are satisfied and* $a_1 b_{11} \neq 0$, *then the plane* $f_4 = a_1 + b_{11}x = 0$, *is an algebraic solution of the LV3 and a first integral* $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ *exists with* $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -2, \lambda_4 = 1$.

(14) *If the conditions* $c_{12} = c''_{31} = b_{13} = b_{23} = b_{33} = e_{31} = e_{32} = 0$ *are satisfied and* $d_{12} d_{21} \neq 0$, *then the plane* $f_4 = (b_{21} - b_{11})x - (b_{12} - b_{22})y = 0$, *is an algebraic solution of the LV3 and a first integral* $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ *exists with* $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 1$.

(15) *If the conditions* $c_{12} = c'_{31} = b_{11} = b_{13} = b_{22} = b_{23} = b_{31} = b_{33} = g_{13} = 0$ *are satisfied and* $b_{12} b_{21} \neq 0$, *then the plane* $f_4 = b_{21}x - b_{12}y = 0$, *is an algebraic solution of the LV3 and a first integral* $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ *exists with* $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1, \lambda_4 = 1$.

Proof. The cofactors of f_1, f_2, f_3 are $K_1 = a_1 + b_{11}x + b_{12}y + b_{13}z$, $K_2 = a_2 + b_{21}x + b_{22}y + b_{23}z$, $K_3 = a_3 + b_{31}x + b_{32}y + b_{33}z$ respectively. Statement (1) use only these three algebraic solutions. For the other statements, it is easy to check, that under suitable assumptions, $f_4 = 0$ is an algebraic solution of the LV3, i.e. that f_4 verifies eq (2.1) where K_4, the cofactor of f_4 takes the values $K_4 = 0$ for statement (1) where $f_4 = 1$, $K_4 = a_1 + b_{11}x + b_{22}y + b_{33}z$ for statement (2), $K_4 = a_1 + b_{11}x + b_{22}y + b_{13}z$ for statements (3), (14) and (15), $K_4 = b_{11}x + b_{22}y$ for statement (4), $K_4 = b_{11}x + b_{22}y + b_{33}z$ for statement (5), $K_4 = b_{11}x$ for statements (7) – (13). We note that f_4 is the algebraic solution of Proposition 1(1) for statement (1), of Proposition 1(4) for statement (2), of Proposition 1(3) for statements (3), (14) and (15) of Proposition 1(6) for statements (4) and (6), of Proposition 1(5) for statement (5), of Proposition 1(8) for statement (7), of Proposition 1(2) for statements (8) – (13). Hence to each statement corresponds a first integral $|x|^{\lambda_1} |y|^{\lambda_2} |z|^{\lambda_3} |f_4|^{\lambda_4}$. ■

We must mention that statements (1)–(5) are, respectively, the invariants I, II, II', III' and III of Cairó and Feix [14] obtained using the Carleman method (with the additional condition $s = 0$ for statements (1), (3) and (5). Note that statements (2) and (3) with $a_1 = a_2 = a_3 = 0$ concerns the ABC system when $b_{11} = b_{22} = b_{33} = 0$ (see for instance Ref. 22). With these additional conditions statement (2) concerns an integrable system (Theorem 6(22)).

Theorem 3. *The LV3 has a first integral* $|f_1|^{\lambda_1} |f_2|^{\lambda_2} |f_3|^{\lambda_3} |f_4|^{\lambda_4} |f_5|^{\lambda_4}$ *formed with the coordinate axes* $f_1 = x$, $f_2 = y$ *and* $f_3 = z$ *and two other algebraic solutions* $f_4 = 0$ *and* $f_5 = 0$ *of degree 1 in the cases described by the following statements*

(1) *If the conditions* $c_{23} = b_{11} = b_{21} = b_{23} = b_{31} = 0$ *are satisfied and* $c_{13} b_{22} b_{33} (a_1 b_{22} b_{33} + a_3(b_{13}(b_{32} - b_{22}) - b_{12} b_{33})) \neq 0$, *then* $f_4 = a_3 + b_{22}y$ *and* $f_5 = (b_{32} - b_{22})y + b_{33}z$ *and* $\lambda_1 = a_3 b_{33}(b_{32} - 2b_{22})$, $\lambda_2 = (b_{32} - 2b_{22})(a_3 b_{13} - a_1 b_{33})$, $\lambda_3 = a_1 b_{22} b_{33} - a_3(b_{12} b_{33} - b_{13} b_{22})$ *and* $\lambda_4 = -a_1 b_{22} b_{33} + a_3(b_{12} b_{33} + b_{13} b_{22} - b_{13} b_{32})$.

(2) *If the conditions* $c_{23} = b_{12} = b_{13} = g_{23} = s'''_{12} = 0$ *are satisfied and* $a_1 b_{11} d_{32} k_{21} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = (b_{32} - b_{22})y - (b_{23} - b_{33})z$ *and* $\lambda_1 = a_2(b_{32} - b_{22})(b_{23} - b_{33})$, $\lambda_2 = a_1 b_{33}(b_{32} - b_{22})$, $\lambda_3 = a_1 b_{22}(b_{23} - b_{33})$, $\lambda_4 = a_1(b_{22}b_{33} - b_{23}b_{32})$.

(3) *If the conditions* $b_{12} = b_{13} = r_{12} = r_{23} = r_{31} = r'''_{23} = 0$ *are satisfied and* $a_1 a_2 a_3 c_{23} b_{11} b_{22} b_{33} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = a_1 a_2 a_3 + a_2 a_3 b_{11} x + a_1 a_3 b_{22} y + a_1 a_2 b_{33} z$ *and* $\lambda_1 = 0$, $\lambda_2 = 2a_3/(a_2 - a_3)$, $\lambda_3 = -2a_2/(a_2 - a_3)$ *and* $\lambda_4 = 1$.

(4) *If the conditions* $c_{23} = g_{23} = h_{23} = p'_{13} = r_{12} = r_{23} = 0$ *are satisfied and* $b_{11} b_{12} b_{13} b_{22} b_{33} \neq 0$, *then* $f_4 = b_{22}y + b_{33}z$ *and* $f_5 = (a_1 + b_{11}x)(b_{21} + b_{11}) + b_{11}b_{22}y + b_{11}b_{33}z$ *and* $\lambda_1 = -(b_{21} + b_{11})/b_{11}$, $\lambda_2 = \lambda_3 = 0$ *and* $\lambda_4 = 1$.

(5) *If the conditions* $c_{23} = b_{12} = b_{13} = b_{22} = d'_{23} = g_{23} = s'_{21} = 0$ *are satisfied and* $a_1 b_{11} b_{23} b_{32} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = b_{32}y + b_{23}z$ *and* $\lambda_1 = a_2/a_1$, $\lambda_2 = -2$, $\lambda_3 = 0$ *and* $\lambda_4 = 1$.

(6) *If the conditions* $a_2 = c''_{31} = b_{12} = b_{13} = b_{22} = b_{31} = d_{21} = d'_{23} = 0$ *are satisfied and* $a_1 a_3 b_{11} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = a_1 + b_{11}x + 2b_{32}y + b_{33}z$ *and* $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = -2$ *and* $\lambda_4 = 1$.

(7) *If the conditions* $a_2 = a_3 = b_{12} = b_{13} = b_{23} = b_{32} = d_{21} = g_{23} = 0$ *are satisfied and* $a_1 b_{11} b_{22} b_{33} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = b_{22}y - b_{33}z$ *and* $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = -1$ *and* $\lambda_4 = 1$.

(8) *If the conditions* $a_2 = a_3 = b_{12} = b_{13} = b_{33} = d'_{32} = d_{21} = g_{23} = 0$ *are satisfied and* $a_1 b_{11} b_{23} b_{32} \neq 0$, *then* $f_4 = a_1 + b_{11}x$ *and* $f_5 = b_{32}y + b_{23}z$ *and* $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$ *and* $\lambda_4 = 1$.

Proof. The first integral can be written as $|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3}|f|^{\lambda_4}$ with $f = f_4 f_5$. The cofactors of f_1, f_2, f_3 are $K_1 = a_1 + b_{11}x + b_{12}y + b_{13}z$, $K_2 = a_2 + b_{21}x + b_{22}y + b_{23}z$, $K_3 = a_3 + b_{31}x + b_{32}y + b_{33}z$ respectively. Now it is easy to check that for statements (1), (2), (5)-(9) f_4 and f_5 are the algebraic solutions of Proposition 1 (2) and (3), for statement (3) f_4 and f_5 are the algebraic solutions of Proposition 1 (2) and (5), for statement (4) f_4 and f_5 are the algebraic solutions of Proposition 1 (3) and (5) and for statement (5) f_4 and f_5 are the algebraic solutions of Proposition 1 (2) and (7). Consequently $f = 0$ is an algebraic solution of the LV3, i.e. f verifies eq (2.1) where K, the cofactor of f takes the value $K = a_3 + 2b_{11}x + 2b_{22}y + 2b_{33}z$ for statement (1), $K = a_3 + (b_{21} + b_{11})x + b_{22}y + b_{33}z$ for statement (2), $K = 2b_{11}x + b_{22}y + 2b_{33}z$ for statement (3), $K = a_1(b_{21} + b_{11})/b_{11} + (b_{21} + b_{11})x + 2b_{22}y + 2b_{33}z$ for statement (4), $K = 2b_{11}x + 2b_{22}y$ for statement (5), $K = a_2 + b_{11}x + 2b_{22}y$ for statement (6), $K = 2b_{11}x + b_{22}y + b_{33}z$ for statement (7) and $K = 2b_{11}x + b_{22}y$ for statement (8). For each statement, a solution of system $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + \lambda_4 K = 0$. Hence to each statement corresponds a first integral $|x|^{\lambda_1}|y|^{\lambda_2}|z|^{\lambda_3}|f_4|^{\lambda_4}|f_5|^{\lambda_4}$. ■

The next two theorems concern LV3 systems having one first integral alone of type $|x|^{\lambda_1}|y|^{\lambda_2}|z|^{\lambda_3}|f_4|^{\lambda_4}$ with f_4 quadratic. These cases are characterized by n conditions among the parameters of the systems. The expression of all the corresponding first integrals are available in the e-mail address: "lcairo@labomath.univ-orleans.fr". To prove each statement one must simply apply (2.1) to obtain f_4 and its corresponding cofactors K_4 and compute the λ_i from the equation $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + \lambda_4 K_4 = 0$, where K_1, K_2, K_3 are the cofactors above mentioned for the coordinate axes, i.e. $K_1 = a_1 + b_{11}x + b_{12}y + b_{13}z$, $K_2 = a_2 + b_{21}x + b_{22}y + b_{23}z$, $K_3 = a_3 + b_{31}x + b_{32}y + b_{33}z$.

5 First integrals with f_4 quadratic

Theorem 4. *Let $f_1 = x$, $f_2 = y$ and $f_3 = z$. If a LV3 has a fourth algebraic solution, $f_4 = 0$ of degree 2 passing through the origin, then a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3}|f_4|^{\lambda_4}$ can be obtained when the conditions given in Table 2 are fulfilled.*

Table 2: First integral conditions of Theorem 4.

st.	$\overrightarrow{\nu}$	n	conditions =0	conditions $\neq 0$
(1)	(0000011111)	4	$c_{12} = c_{31} = n'_{23} = g_{32} = 0$	$d_{12}d_{13}d_{23}j_{13}$
(2)	(0011000111)	4	$c_{23} = b_{31} = b_{21} = n_{23} = 0$	$b_{22}b_{33}d_{23}d''_{23}d_{32}d''_{32}$
(3)	(0001010010)	5	$v'_{123} = b_{23} = i_{32} = g_{31} = k_{21} = 0$	([1])
(4)	(0001101001)	5	$c'_{31} = b_{12} = b_{22} = b_{32} = n_{31} = 0$	([2])
(5)	(0000111111)	5	$a_1 = a_2 = a_3 = n'_{12} = n'_{23} = 0$	([3])
(6)	(0000001111)	5	$a_1 = a_2 = a_3 = i'''_{23} = n'_{23} = 0$	([4])
(7)	(0011001111)	6	$c_{32} = b_{21} = e_{31} = n'_{32} = r_{13} = r_{23} = 0$	$a_1a_2a_3c''_{12}b_{22}b_{33}$
(8)	(0011011001)	6	$c_{23} = b_{12} = b_{13} = d'_{32} = g_{23} = r_{31} = 0$	$b_{11}d''_{23}d_{21}$
(9)	(0011001100)	6	$c_{23} = b_{13} = b_{21} = d'_{23} = e_{31} = j_{31} = 0$	$a_1a_2a_3c_{12}b_{12}b_{33}$
(10)	(0011010110)	6	$c_{23} = b_{21} = b_{23} = d_{13} = d_{31} = r_{12} = 0$	$b_{13}b_{22}d_{12}$
(11)	(0011001110)	6	$c_{32} = b_{23} = b_{21} = b_{13} = e_{31} = s''_{12} = 0$	$a_1a_2a_3d_{32}$
(12)	(0011111111)	6	$c'_{21} = c_{23} = n_{12} = r_{23} = n_{31} = g_{32} = 0$	$b_{11}b_{22}b_{33}d_{23}d_{32}d''_{31}$
(13)	(0001111111)	6	$c_{12} = c'_{32} = d_{12} = n_{23} = n_{31} = d_{21} = 0$	([5])
(14)	(0001111111)	6	$c_{12} = c'_{32} = n_{23} = n_{31} = n'_{21} = n'_{13} = 0$	$b_{12}b_{13}b_{33}d''_{23}d''_{31}i_{31}$
(15)	(0001011111)	6	$c_{12} = c'_{32} = o_{123} = n_{23} = r_{31} = g_{32} = 0$	$b_{21}b_{22}b_{33}d'_{13}d''_{23}$
(16)	(0011101001)	6	$c'_{21} = c_{23} = d'_{12} = g_{31} = r_{21} = n_{31} = 0$	$b_{33}d''_{13}d''_{23}g'_{21}$
(17)	(0011110100)	6	$c'_{21} = c_{23} = d''_{31} = d'_{13} = g_{21} = n_{12} = 0$	$b_{22}d''_{21}d''_{32}g'_{31}$
(18)	(0011111111)	6	$c_{23} = c'_{21} = n_{12} = n_{31} = o_{123} = n_{23} = 0$	$b_{22}b_{33}d_{32}d''_{32}d''_{13}$
(19)	(0001010111)	6	$c_{12} = c'_{31} = b_{31} = e_{21} = n_{23} = n'_{32} = 0$	$b_{12}b_{13}d_{12}i_{31}$
(20)	(0001011110)	6	$c_{12} = c'_{31} = b_{13} = b_{23} = g_{32} = n'''_{12} = 0$	$b_{11}b_{33}d_{21}d_{32}d''_{32}$
(21)	(0001111001)	6	$c_{12} = c'_{31} = b_{13} = b_{23} = g_{31} = n^{iv}_{12} = 0$	$b_{22}d_{21}d_{31}d''_{31}$
(22)	(0001110110)	6	$c_{12} = c'_{31} = b_{23} = d''_{31} = d'_{13} = n'_{12} = 0$	$b_{13}d_{32}d''_{32}g'_{31}$
(23)	(0001111100)	6	$c_{12} = c'_{31} = b_{13} = d'_{23} = n'_{12} = r_{32} = 0$	$b_{12}d_{31}d''_{31}g'_{32}$
(24)	(0111000011)	6	$c_{12} = c_{23} = b_{32} = d'_{21} = d'_{31} = n''_{23} = 0$	$a_1a_2a_3d''_{13}d_{23}$
(25)	(0001010110)	6	$c_{12} = c'_{32} = b_{23} = d_{12} = d_{21} = g'_{32} = 0$	$b_{13}b_{33}d_{13}d''_{32}$
(26)	(0001011100)	6	$c_{12} = c'_{31} = b_{13} = d'_{23} = d''_{32} = g_{32} = 0$	$b_{11}b_{12}b_{21}d_{21}$
(27)	(0011101000)	6	$c'_{21} = c_{23} = b_{13} = d'_{12} = d''_{21} = g_{31} = 0$	$b_{23}b_{33}d_{23}d''_{31}$
(28)	(0011111000)	6	$c'_{21} = c_{23} = b_{12} = b_{13} = g_{32} = r_{23} = 0$	$b_{22}b_{33}d''_{31}d_{32}$
(29)	(0011111000)	6	$c'_{21} = c_{23} = b_{12} = b_{13} = d_{32} = d_{23} = 0$	$d_{21}d''_{21}d_{31}d''_{31}g_{23}$
(30)	(0001001111)	6	$a_1 = c'_{32} = b_{11} = n_{23} = n'_{32} = g'_{32} = 0$	$b_{12}b_{13}b_{33}d_{13}d''_{32}$
(31)	(0001000111)	6	$a_1 = c'_{32} = b_{11} = b_{21} = b_{31} = n_{32} = 0$	([6])
(32)	(0001010011)	6	$a_3 = c''_{12} = b_{31} = b_{33} = e_{21} = g_{31} = 0$	$b_{12}b_{22}b_{23}d_{12}g''_{12}$
(33)	(0001000111)	6	$c_{12} = c'_{32} = b_{11} = b_{21} = b_{31} = n_{23} = 0$	([7])
(34)	(0001101000)	6	$c_{32} = c'_{21} = b_{12} = b_{13} = b_{22} = b_{32} = 0$	([8])
(35)	(0001111111)	7	$c_{12} = c'_{32} = b_{12} = b_{22} = d_{21} = d''_{23} = n_{31} = 0$	$b_{11}b_{33}b_{32}d''_{13}$
(36)	(0011010000)	7	$c_{23} = b_{12} = b_{31} = b_{32} = b_{33} = e_{21} = g''_{21} = 0$	a_1a_2
(37)	(0111111111)	7	$c_{12} = c_{23} = r'_{23} = r_{31} = r_{21} = r'_{32} = n'_{13} = 0$	$a_1a_2a_3b_{11}b_{22}b_{33}$
(38)	(0011111111)	7	$c_{23} = c'_{21} = n_{12} = r_{23} = n_{31} = n'_{21} = n'_{13} = 0$	([9])
(39)	(0111001111)	7	$c_{12} = c_{13} = b_{31} = d'_{21} = n'_{23} = n''_{13} = r_{32} = 0$	$b_{13}b_{22}b_{32}e'_{32}j_{31}$
(40)	(0001011111)	7	$c_{12} = c'_{32} = b_{22} = o_{123} = n_{23} = r_{31} = g_{32} = 0$	$b_{32}b_{33}g'_{31}$
(41)	(0011101011)	7	$c'_{21} = c'_{31} = b_{32} = d'_{12} = n_{13} = n'_{31} = r_{21} = 0$	$b_{11}b_{12}b_{13}b_{23}b_{31}d''_{31}$
(42)	(0011111011)	7	$c'_{21} = c_{32} = b_{12} = b_{32} = o_{123} = n_{31} = i''_{23} = 0$	$b_{33}d''_{13}d''_{21}$
(43)	(0011111011)	7	$c'_{21} = c_{32} = b_{12} = b_{32} = g_{23} = n_{31} = n'_{13} = 0$	$b_{22}b_{31}b_{33}d''_{31}$

Table 2: First integral conditions of Theorem 4 (continued).

st.	$\overrightarrow{\nu}$	n	conditions =0	conditions $\neq 0$
(44)	(0111001111)	7	$c_{12}=c_{13}=b_{31}=b_{32}=d'_{21}=n_{23}=n'_{32}=0$	$b_{12}b_{13}b_{22}b_{33}d''_{12}$
(45)	(0001111111)	7	$c_{12}=c'_{32}=b_{11}=b_{21}=d_{12}=n_{23}=d''_{13}=0$	$b_{31}b_{33}d''_{23}$
(46)	(0111001110)	7	$c_{12}=c_{32}=b_{23}=b_{31}=b_{13}=d'_{21}=j''_{13}=0$	$a_1b_{11}b_{22}b_{32}b_{33}$
(47)	(0011111010)	7	$c'_{21}=c'_{31}=b_{12}=b_{23}=b_{32}=b_{13}=j'_{23}=0$	$b_{22}d_{31}d''_{21}d''_{31}$
(48)	(0001000111)	7	$c_{12}=c'_{32}=b_{11}=b_{21}=b_{31}=b_{22}=d''_{23}=0$	([10])
(49)	(0111011011)	7	$c_{13}=c_{12}=b_{12}=b_{31}=b_{21}=b_{32}=n''_{23}=0$	$b_{11}b_{33}d_{23}$
(50)	(0010001010)	7	$a_1=c_{23}=b_{11}=b_{32}=d_{23}=d_{13}=g_{23}=0$	$b_{22}d_{12}$
(51)	(0001011000)	7	$a_2=c_{13}=b_{13}=b_{22}=b_{33}=g_{32}=g_{31}=0$	$b_{11}b_{23}d_{21}$
(52)	(0001000111)	7	$a_1=c'_{32}=b_{11}=b_{21}=b_{22}=b_{31}=d''_{23}=0$	([11])
(53)	(0011001111)	7	$a_2=a_3=b_{21}=b_{22}=b_{33}=e_{31}=n'_{23}=0$	$b_{11}b_{13}$
(54)	(0001101011)	7	$a_1=a_2=a_3=n_{13}=n'_{13}=q_{12}=q''_{21}=0$	([12])
(55)	(0111001111)	8	$c_{31}=c_{21}=b_{13}=b_{31}=d'_{21}=e'_{32}=n'_{12}=n'_{23}=0$	$b_{22}b_{33}$
(56)	(0011101000)	8	$c_{23}=c'_{31}=b_{13}=b_{33}=d_{31}=d'_{12}=g_{31}=g'_{23}=0$	$a_1b_{11}b_{22}b_{23}$
(57)	(0011110100)	8	$c'_{21}=c_{23}=b_{13}=b_{23}=b_{33}=d''_{31}=d''_{32}=n_{12}=0$	$b_{22}d''_{21}$
(58)	(0001111110)	8	$c_{12}=c'_{32}=b_{11}=b_{13}=b_{22}=b_{23}=g'_{23}=g'''_{13}=0$	$b_{21}b_{33}$
(59)	(0011001100)	8	$a_1=c_{23}=b_{11}=b_{13}=b_{21}=b_{31}=d'_{23}=j_{13}=0$	$a_2d''_{32}$
(60)	(0001101011)	8	$a_1=a_2=a_3=b_{11}=d''_{13}=n'_{13}=n'''_{23}=p''_{23}=0$	([13])
(61)	(0001111100)	8	$a_1=a_2=a_3=b_{13}=d'_{23}=d''_{31}=d''_{32}=n'_{12}=0$	$b_{11}b_{22}b_{12}b_{21}$
(62)	(0111010111)	8	$a_1=a_2=a_3=b_{21}=d'_{31}=j'_{21}=n_{23}=n'_{23}=0$	$b_{22}b_{33}d''_{13}(d''_{13}-4b_{33})$
(63)	(0001101011)	8	$a_1=a_2=a_3=b_{22}=b_{33}=n_{31}=n'_{31}=g'_{31}=0$	$b_{21}b_{23}$

([1]) $a_1a_2d_{13}(d_{13}k_{12}-a_2b_{13}b_{22})\neq 0$

([2]) $b_{11}b_{33}(a_1(2b_{21}b_{33}^2-5b_{13}b_{21}b_{33}+2b_{13}^2b_{21}+b_{11}b_{23}b_{33})-2a_2b_{11}d_{13}^2)\neq 0$

([3]) $d_{13}d_{31}j_{23}({b_{11}}^3(2\,b_{32}\,b_{13}+b_{22}\,b_{33}-3\,b_{33}\,b_{32})-b_{22}\,{b_{31}}^3\,d_{13}-{b_{11}}^2b_{21}(b_{32}\,b_{13}+2\,b_{22}\,b_{33}-3\,b_{33}\,b_{32})+b_{13}\,{b_{21}}^2\,b_{31}\,d_{32}+b_{21}{b_{31}}^2\,d_{13}\,d''_{32}+{b_{11}}^2b_{31}(3\,b_{22}\,b_{33}+2\,b_{33}\,b_{32}-4\,b_{22}\,b_{13}-b_{32}\,b_{13})-b_{11}b_{33}{b_{21}}^2\,d_{32}+4\,b_{11}\,b_{22}{b_{31}}^2\,d_{13}+2\,b_{11}\,b_{21}\,b_{31}(2\,b_{13}\,b_{22}-b_{22}\,b_{33}-b_{13}\,b_{32}))\neq 0$

([4]) $d_{13}j_{13}(4\,b_{11}\,b_{23}\,d_{32}^2-{b_{12}}^2b_{23}\,d_{31}-b_{12}\,b_{23}b_{31}\,(3\,b_{32}-4\,b_{22})+b_{11}\,b_{12}\,b_{23}\,(3\,b_{32}-4\,b_{22})+b_{12}\,b_{13}\,b_{22}\,b_{31}-b_{11}\,b_{13}(2\,{b_{32}}^2-b_{32}\,b_{22}-2\,{b_{22}}^2)-b_{11}\,b_{12}\,b_{13}\,b_{22}-b_{13}\,b_{31}d''_{32}(2\,b_{32}-3\,b_{22}))\neq 0$

([5]) $b_{21}b_{33}d_{23}d''_{13}d''_{31}d''_{23}d''_{32}g_{21}\neq 0$

([6]) $b_{22}b_{33}d''_{23}(b_{12}d'_{23}d''_{23}+b_{22}b_{13}b_{33})\neq 0$

([7]) $a_1a_2a_3d''_{23}(b_{12}d'_{23}d''_{23}+b_{22}(b_{13}b_{33}-2d_{23}^2))\neq 0$

([8]) $b_{33}d''_{31}(b_{33}d''_{21}-b_{23}d''_{31})\neq 0$

([9]) $b_{22}b_{33}d''_{12}d''_{13}d''_{21}(2b_{12}-3b_{22})\neq 0$

([10]) $a_1a_2a_3b_{33}(3b_{12}b_{33}+b_{32}d''_{13})\neq 0$

([11]) $b_{23}b_{32}(b_{13}b_{32}+3b_{12}b_{33})\neq 0$

([12]) $b_{11}b_{33}d''_{13}(b_{33}b_{32}^2+(4b_{13}-5b_{33})b_{22}^2)\neq 0$

([13]) $b_{21}b_{23}b_{33}(b_{23}^2-b_{33}d_{23})\neq 0$

Theorem 5. *Let $f_1=x$, $f_2=y$ and $f_3=z$. If a LV3 has a fourth algebraic solution, $f_4=0$ of degree 2 not passing through the origin, then a first integral $|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3}|f_4|^{\lambda_4}$ can be obtained when the conditions given in Table 3 are fulfilled.*

Table 3: First integral conditions of Theorem 5.

st.	$\overrightarrow{\nu}$	n	conditions =0	conditions $\neq 0$
(1)	(1011000011)	4	$r'_{23}=b_{31}=b_{21}=b_{32}=0$	$a_2a_3c''_{23}$
(2)	(1110110100)	5	$b_{13}=b_{23}=b_{33}=r'_{12}=r'_{21}=0$	(*)
(3)	(1110100100)	5	$c''_{12}=b_{13}=b_{23}=d_{12}=d_{21}=0$	$\|a_1\|+\|b_{11}\|+\|b_{22}\|$

Table 3: First integral conditions of Theorem 5 (continued).

st.	$\overrightarrow{\nu}$	n	conditions =0	conditions $\neq 0$
(4)	(1001000111)	5	$a_2 = b_{21} = b_{22} = b_{31} = d''_{23} = 0$	a_3
(5)	(1111111111)	6	$k_{12} = r_{12} = r'_{23} = r'_{31} = r'_{32} = r_{13} = 0$	$a_1a_2c'''_{23}c'''_{32}$
(6)	(1111111111)	6	$k_{23} = k_{32} = r'_{12} = r'_{31} = r'_{21} = r'_{13} = 0$	$a_1a_2a_3c_{23}b_{11}b_{22}b_{33}$
(7)	(1111111111)	6	$k_{23} = k_{32} = r'_{12} = r_{31} = r'_{21} = r'_{13} = 0$	$a_1a_2a_3b_{13}b_{22}$
(8)	(1111111111)	6	$k_{31} = k_{13} = k'_{32} = r'_{12} = r'_{23} = r'_{21} = 0$	$a_1a_2a_3$
(9)	(1011011111)	6	$e_{21} = e_{31} = r'_{12} = r_{23} = k_{32} = r'_{13} = 0$	$a_2a_3c_{23}c''_{12}c''_{13}$
(10)	(1111000111)	6	$d'_{31} = d'_{21} = r'_{12} = r'_{23} = r'_{32} = r'_{13} = 0$	$a_1a_2a_3v'_{231}$
(11)	(1001001011)	6	$e_{31} = e_{32} = g''_{31} = r_{12} = r'_{13} = r'_{23} = 0$	$a_3c_{12}c''_{13}c''_{23}$
(12)	(1111001111)	6	$b_{31} = d'_{21} = k_{23} = k_{32} = r'_{12} = r'_{13} = 0$	$a_1a_2a_3c''_{13}$
(13)	(1011001111)	6	$b_{21} = e_{31} = r'_{23} = n'_{23} = r'_{32} = r'_{13} =0$	$a_2a_3c''_{13}c'''_{13}v'_{132}$
(14)	(1111000011)	6	$b_{32} = d'_{31} = d'_{21} = r_{12} = r_{23} = r'_{13} = 0$	$a_1a_2a_3c'_{12}$
(15)	(1011001001)	6	$b_{21} = d'_{12} = r'_{23} = e_{31} = d'_{32} = r'_{13} = 0$	$a_2a_3c''_{13}v'_{312}$
(16)	(1011001111)	6	$b_{21} = e_{31} = r_{23} = r'_{12} = r'_{13} = k_{23} = 0$	$a_2a_3c''_{13}$
(17)	(1111001001)	6	$b_{31} = r_{12} = r'_{23} = r_{13} = d'_{12} = g_{31} = 0$	$a_1a_2a_3c'_{21}c'''_{21}$
(18)	(1011010110)	6	$b_{23} = b_{31} = e_{21} = r'_{12} = r_{32} = s'_{13} = 0$	$a_2a_3c''_{12}w''_{312}$
(19)	(1011001011)	6	$b_{21} = b_{32} = e_{31} = o_{123} = r_{23} = r'_{13} = 0$	$a_2a_3b_{13}c''_{13}w''_{213}$
(20)	(1111001111)	6	$b_{31} = b_{32} = r_{31} = r_{32} = r'_{12} = d'_{21} = 0$	$a_1a_2a_3b_{11}b_{22}b_{33}$
(21)	(1011001010)	6	$b_{13} = b_{21} = d'_{23} = e_{31} = m_{12} = r'_{32} = 0$	$a_1a_2a_3c''_{13}v''_{132}$
(22)	(1011011000)	6	$b_{12} = b_{13} = d_{23} = d_{32} = e_{21} = e_{31} = 0$	$a_2a_3c_{23}c''_{12}c''_{13}$
(23)	(1111001011)	6	$b_{31} = b_{32} = d_{12} = d_{21} = r_{13} = r'_{23} = 0$	$a_1a_3a_2c''_{23}$
(24)	(1111001011)	6	$b_{31} = b_{32} = r_{12} = r_{23} = r_{13} = r'''_{12} = 0$	$a_1a_3a_2b_{11}b_{22}c_{12}$
(25)	(1011010110)	6	$b_{31} = b_{23} = r'_{12} = e_{21} = r''_{32} = s'_{13} = 0$	$a_2a_3c_{12}c'''_{23}v''_{123}$
(26)	(1011001010)	6	$b_{23} = b_{21} = b_{32} = b_{13} = e_{31} = s_{12} = 0$	$a_2a_3c''_{13}$
(27)	(1011011111)	6	$c_{32} = r'_{12} = r_{23} = e_{31} = e_{21} = r'_{13} = 0$	$a_2a_3c''_{12}c''_{13}c'''_{12}c'''_{13}d_{32}$
(28)	(1001010011)	6	$c''_{12} = b_{31} = r'_{23} = e_{12} = e_{21} = e_{32} = 0$	$a_3c_{13}c'_{13}g''_{12}k_{13}$
(29)	(1001011001)	6	$c''_{12} = b_{32} = e_{12} = e_{21} = e_{31} = r'_{13} = 0$	$a_3c''_{13}g''_{12}m_{23}$
(30)	(1001010010)	6	$c''_{12} = b_{23} = b_{31} = e_{12} = e_{21} = e_{32} = 0$	$a_3b_{13}d_{13}c_{13}$
(31)	(1011001111)	6	$a_2 = b_{22} = b_{21} = e_{31} = n'_{32} = r'_{13} = 0$	$a_3b_{13}b_{23}c''_{13}$
(32)	(1111000111)	7	$c'''_{23} = d'_{21} = d'_{31} = e'_{32} = e''_{23} = r'_{12} = r'_{13} = 0$	$a_1a_2a_3c'''_{12}$
(33)	(1111000111)	7	$c'''_{13} = b_{13} = d'_{21} = d'_{31} = k_{23} = k_{32} = r'_{12} = 0$	$a_1a_2a_3c'''_{12}$
(34)	(1111011111)	7	$c'''_{21} = b_{21} = b_{31} = r'_{12} = r_{31} = r'_{23} = n_{23} = 0$	$a_1a_2a_3$
(35)	(1111001001)	7	$c'''_{21} = b_{21} = b_{31} = r_{13} = d'_{12} = g_{31} = r'_{23} = 0$	$a_1a_2a_3$
(36)	(1111000111)	7	$c''_{23} = d'_{21} = e_{23} = d'_{31} = e_{32} = r'_{12} = r'_{13} = 0$	$a_1a_2a_3$
(37)	(1111001100)	7	$w_{231} = b_{13} = b_{31} = d'_{21} = d'_{23} = r'_{12} = r'_{32} = 0$	$a_1a_3c''_{31}$
(38)	(1011010110)	7	$w_{312} = b_{13} = b_{23} = b_{31} = e_{21} = n'_{23} = r_{23} = 0$	$a_2c''_{12}$
(39)	(1111111111)	7	$v_{123} = r'_{12} = r'_{23} = r'_{31} = r'_{21} = r'_{32} = r'_{13} = 0$	$a_1a_2a_3b_{11}b_{22}b_{33}$
(40)	(1111011010)	7	$v_{123} = b_{12} = b_{23} = b_{31} = b_{21} = b_{32} = b_{13} = 0$	$a_1a_2c''_{12}$
(41)	(1001011011)	7	$c''_{12} = e_{31} = e_{32} = g_{13} = g_{23} = r'_{13} = r'_{23} = 0$	$c_{13}c''_{13}c''_{13}$
(42)	(1111011111)	7	$c'''_{31} = b_{31} = b_{21} = r_{12} = r'_{23} = r'_{32} = r'_{13} = 0$	$a_1a_2a_3$
(43)	(1111001110)	7	$c'''_{32} = b_{23} = b_{31} = b_{32} = b_{13} = r'_{12} = d'_{21} = 0$	$a_1a_2a_3c_{13}$
(44)	(1011011011)	7	$c_{23} = b_{12} = b_{32} = e_{31} = e_{21} = r'_{13} = r'_{23} = 0$	$a_1a_2a_3c''_{12}$
(45)	(1011011111)	7	$w_{123} = r'_{12} = r'_{23} = e_{31} = e_{21} = r'_{32} = r'_{13} = 0$	$a_2a_3c_{23}c''_{12}c''_{13}$
(46)	(1111011111)	7	$w_{123} = b_{31} = b_{21} = r_{12} = r'_{23} = r'_{32} = r_{13} = 0$	$a_1a_2a_3b_{23}b_{32}c''_{32}$
(47)	(1011011010)	7	$w_{123} = b_{12} = b_{23} = b_{32} = b_{13} = e_{31} = g_{23} = 0$	$a_2a_3c''_{12}c''_{31}c'''_{31}$
(48)	(1111010111)	7	$w'''_{231} = b_{21} = d'_{31} = r_{12} = r'_{23} = r'_{32} = r'_{13} = 0$	$a_1a_2a_3$
(49)	(1111001110)	7	$w'''_{123} = b_{23} = b_{31} = b_{13} = d'_{21} = r'_{12} = r_{32} = 0$	$a_1a_2a_3c'''_{12}$
(50)	(1011011011)	7	$w'_{213} = b_{12} = b_{32} = e_{31} = e_{21} = r_{23} = r'_{13} = 0$	$a_3c_{13}c''_{13}$
(51)	(1011111111)	7	$a_1 = b_{11} = n_{12} = n_{31} = n'_{23} = n'_{31} = k_{23} = 0$	a_2a_3
(52)	(1011111111)	7	$a_1 = b_{11} = d''_{12} = n_{31} = n_{23} = l'_{23} = r'_{23} = 0$	a_2a_3

Table 3: First integral conditions of Theorem 5 (continued).

st.	$\overrightarrow{\nu}$	n	conditions =0	conditions $\neq 0$
(53)	(1111000111)	7	$a_3 = b_{33} = d'_{21} = d'_{31} = d''_{32} = l_{12} = r'_{12} = 0$	$a_1a_2c_{12}$
(54)	(1111000111)	7	$a_2 = b_{22} = d'_{21} = d'_{31} = l_{13} = r'_{13} = r'_{23} = 0$	$a_1a_3c_{13}c'''_{13}$
(55)	(1011001001)	7	$a_3 = b_{21} = b_{33} = d'_{12} = e_{31} = g_{31} = l_{21} = 0$	$a_1a_2c_{12}$
(56)	(1111111111)	7	$a_1 = b_{11} = b_{13} = r'_{12} = r'_{23} = r_{32} = l_{23} =0$	a_2a_3
(57)	(1111111111)	7	$a_1 = b_{11} = b_{13} = r'_{12} = r'_{23} = r'_{32} = l_{23} =0$	a_2a_3
(58)	(1111111111)	7	$a_1 = b_{11} = b_{12} = k'_{23} = r_{32} = r'_{13} = l_{23} =0$	a_2a_3
(59)	(1011011111)	7	$a_2 = b_{22} = b_{23} = e_{31} = g_{23} = l_{13} = n'_{23} = 0$	$a_1a_3c''_{13}$
(60)	(1011010111)	7	$a_3 = b_{31} = b_{33} = e_{21} = d''_{32} = r'_{12} = r^{iv}_{12} = 0$	$a_2c''_{12}c'''_{12}$
(61)	(1011001111)	7	$a_3 = b_{21} = b_{33} = e_{31} = d''_{32} = l_{12} = k'_{12} = 0$	$a_1a_2c_{12}$
(62)	(1111000111)	7	$a_2 = b_{22} = b_{23} = l_{13} = d'_{21} = d'_{31} = r'_{13} = 0$	a_1a_3
(63)	(1111000111)	7	$a_2 = b_{22} = b_{32} = d'_{21} = d'_{31} = r'_{13} = o_{123} = 0$	a_1a_3
(64)	(1111000110)	7	$a_2 = b_{22} = b_{23} = d'_{21} = g_{32} = l_{31} = r_{31} = 0$	$a_1a_3c'_{13}$
(65)	(1111001001)	7	$a_1 = b_{11} = b_{31} = d'_{12} = d_{13} = g_{31} = r'_{23} = 0$	a_2a_3
(66)	(1111000011)	7	$a_3 = b_{32} = b_{33} = d'_{21} = d'_{31} = l_{12} = r_{12} = 0$	$a_1a_2c'_{12}$
(67)	(1111001111)	7	$a_2 = b_{22} = b_{31} = b_{32} = d'_{21} = d_{23} = r_{13} = 0$	$a_1a_3c''_{13}$
(68)	(1111001111)	7	$a_2 = b_{22} = b_{23} = b_{31} = d'_{21} = n'_{23} = r'_{13} = 0$	$a_1a_3c''_{13}$
(69)	(1111010111)	7	$a_3 = b_{21} = b_{32} = b_{33} = d'_{31} = l_{12} = r'_{12} = 0$	$a_1a_2c''_{12}$
(70)	(1011010111)	7	$a_3 = b_{13} = b_{31} = b_{33} = e_{21} = r'_{12} = n'_{23} = 0$	$a_1a_2c''_{12}$
(71)	(1011001111)	7	$a_3 = b_{33} = b_{21} = b_{32} = e_{31} = n'_{23} = l_{12} = 0$	a_1a_2
(72)	(1011010110)	7	$a_3 = b_{23} = b_{31} = b_{33} = e_{21} = d_{32} = r'_{12} = 0$	$a_2c''_{12}$
(73)	(1011001011)	7	$a_3 = b_{21} = b_{32} = b_{33} = e_{31} = l_{12} = o_{123} = 0$	$a_1a_2c'_{12}$
(74)	(1011010111)	7	$a_2 = a_3 = b_{22} = b_{31} = b_{33} = e_{21} = n'_{23} = 0$	$b_{12}g''_{12}$
(75)	(1010011100)	8	$c''_{13} = b_{13} = b_{23} = b_{33} = e_{21} = e_{31} = k_{32} = r'_{12} = 0$	a_2c_{32}
(76)	(1001011001)	8	$c''_{12} = b_{12} = b_{22} = b_{32} = e_{31} = g_{32} = k_{23} = r'_{13} = 0$	$a_3c''_{13}$
(77)	(1001011011)	8	$c''_{12} = c'''_{13} = b_{13} = e_{31} = e_{32} = g_{31} = g_{32} = r'_{23} = 0$	$a_1a_2a_3$
(78)	(1111010111)	8	$c'''_{12} = c''_{32} = b_{21} = b_{23} = b_{32} = d'_{31} = e_{12} = r'_{13} = 0$	$a_1a_2a_3$
(79)	(1111010111)	8	$c''_{12} = c_{13} = b_{12} = b_{21} = b_{23} = b_{32} = d'_{31} = r'_{13} = 0$	$a_1a_2a_3$
(80)	(1011100110)	8	$a_1 = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d'_{13} = r'_{32} = 0$	$a_2c''_{23}b_{22}b_{33}$
(81)	(1011011111)	8	$a_2 = c'''_{31} = b_{22} = d''_{23} = e_{31} = g_{23} = r'_{13} = o_{123} = 0$	a_1a_3
(82)	(1011011111)	8	$a_3 = c'''_{21} = b_{33} = e_{31} = g_{23} = n_{23} = r'_{12} = o_{123} = 0$	a_1a_2
(83)	(1111001111)	8	$a_2 = c'''_{31} = b_{22} = b_{31} = d'_{21} = d'_{13} = g'''_{31} = d''_{23} = 0$	a_1a_3
(84)	(1111010111)	8	$a_3 = c'''_{21} = b_{21} = b_{33} = d'_{31} = d''_{32} = g'''_{21} = r_{12} = 0$	a_1a_2
(85)	(1011011011)	8	$a_3 = c'''_{12} = b_{12} = b_{32} = b_{33} = e_{31} = g_{23} = g'''_{12} = 0$	a_1a_2
(86)	(1111000011)	8	$a_3 = c'''_{12} = b_{12} = b_{32} = b_{33} = d'_{21} = d'_{31} = g'''_{12} = 0$	a_1a_2

$(^*)$ $a_1a_2c'''_{12}c'''_{21}(a_3{c''_{12}}^2b_{12}b_{21} - a_1^2c'''_{12}b_{21}b_{32} - a_2^2c'''_{21}b_{12}b_{31}) \neq 0$

6 Integrability

We say that a 3-dimensional system like (1.1) is *integrable* if it has two independent first integrals. Theorems 6–10 concern integrable LV3 having at least one first integral formed with linear algebraic solutions and Theorem 11 is for integrable cases with algebraic solutions being both quadratic. In these theorems we indicate the nature of the second first integral computed using the notation P1(st) and Tk(st), respectively for Proposition 1, statement st and Theorem k, statement st. and Q is for quadratic first integrals which do not belong to the statements of Theorems 4 and 5.

Theorem 6. *An LV3 is integrable with two first integrals one of which being of the type of Theorem 2(1) in the cases of Table 4.*

Table 4: Integrability conditions of Theorem 6.

st.	n	conditions =0	conditions $\neq 0$	f.i. type
(1)	4	$r_{12} = r_{23} = r_{31} = a_1 b_{22} d_{23} + a_2 h_{32} = 0$	$a_1 a_2 a_3 b_{11} b_{22} b_{33} k_{12}$	P1(5)
(2)	5	$b_{13} = b_{23} = b_{33} = r_{12} = a_3 b_{22} d_{21} + a_2 h_{12} = 0$	$a_1 a_2 b_{11} b_{22}$	T2(4)
(3)	5	$a_3 = b_{33} = r_{12} = l_{12} = o_{123} = 0$	$a_1 a_2 b_{11} b_{22}$	T2(2)
(4)	6	$c_{12} = c_{23} = n_{12} = n_{23} = n_{31} = o_{123} = 0$	$b_{12} d_{21} d''_{31}$	Q
(5)	6	$b_{21} = b_{31} = k_{23} = k_{32} = r'_{12} = r'_{13}$	$a_1 a_2 c''_{12} c''_{13}$	Q
(6)	6	$b_{31} = b_{32} = d_{12} = d_{21} = r'_{23} = r'_{13}$	$a_1 a_2$	Q
(7)	6	$a_1 = c_{23} = b_{11} = n_{23} = n'_{23} = g_{32} = 0$	$b_{13} d_{13} d_{23} d''_{23} j_{12}$	Q
(8)	6	$a_3 = b_{33} = b_{31} = b_{32} = r_{12} = o_{123} = 0$	$a_1 a_2 k_{12}$	P1(7)
(9)	7	$a_2 = b_{22} = b_{21} = b_{23} = b_{31} = l_{13} = n'_{23} = 0$	$a_1 a_3 c''_{13}$	Q
(10)	7	$c_{23} = b_{12} = b_{22} = b_{32} = d_{23} = g_{23} = r_{31} = 0$	$a_1 b_{33} k_{13}$	T2(4)
(11)	7	$a_2 = a_3 = b_{22} = b_{23} = b_{32} = b_{33} = g_{23} = 0$	$b_{11} b_{21}$	P1(3,7)
(12)	7	$a_3 = a_1 = b_{33} = b_{31} = b_{12} = b_{13} = b_{11} = 0$	$b_{32} d_{32}$	P1(7)
(13)	7	$a_1 = a_2 = b_{11} = b_{12} = b_{21} = b_{22} = d''_{13} = 0$	$a_3 b_{33} d_{23}$	Q
(14)	7	$a_1 = a_2 = b_{11} = b_{12} = b_{21} = b_{22} = j_{12} = 0$	$a_3 d_{23} d''_{23}$	Q
(15)	7	$a_2 = a_3 = b_{23} = b_{22} = b_{33} = d''_{32} = i'''_{23} = 0$	$b_{11} b_{13} d_{31}$	Q
(16)	7	$a_1 = a_3 = b_{11} = b_{13} = b_{31} = b_{33} = i'''_{13} = 0$	$a_2 b_{32} g_{13}$	Q
(17)	7	$a_2 = a_3 = b_{22} = b_{23} = b_{32} = b_{33} = i_{32} = 0$	$b_{11} b_{12} b_{21} d_{21}$	Q
(18)	7	$a_1 = a_2 = a_3 = b_{33} = e_{31} = e_{32} = o_{123} = 0$	$b_{11} b_{22} b_{23} b_{32}$	T2(2)
(19)	7	$a_1 = a_2 = a_3 = b_{13} = d''_{31} = i_{31} = o_{123} = 0$	$b_{11} b_{33} d_{12} d_{21}$	Q
(20)	7	$a_1 = a_2 = a_3 = b_{13} = b_{31} = i'_{31} = o_{123} = 0$	$b_{11} b_{32}$	Q
(21)	7	$a_1 = a_2 = a_3 = b_{31} = b_{32} = b_{33} = o_{123} = 0$	$b_{11} d_{21} (b_{11} b_{22} - b_{12} b_{21})$	P1(4)
(22)	7	$a_1 = a_2 = a_3 = b_{11} = b_{22} = b_{33} = o_{123} = 0$		T2(2)
(23)	7	$a_1 = a_2 = a_3 = b_{12} = b_{13} = b_{22} = b_{32} = 0$	$b_{11} d_{31}$	P1(3)
(24)	8	$c_{23} = g_{23} = d_{12} = d_{21} = d_{32} = b_{13} = b_{23} = b_{33} = 0$	$a_1 a_2 a_3 c_{12}$	T2(5)
(25)	8	$c_{12} = c'_{32} = d_{12} = d_{21} = d_{13} = d_{31} = d_{23} = d_{32} = 0$	b_{13}	T4(15)
(26)	8	$c'_{31} = b_{12} = b_{13} = b_{22} = b_{23} = b_{32} = b_{33} = d_{31} = 0$	l_{32}	T2(5)
(27)	8	$c_{23} = b_{12} = b_{13} = b_{22} = b_{23} = b_{32} = b_{33} = g_{23} = 0$	$a_1 b_{11} k_{21}$	T2(7)
(28)	8	$c''_{31} = b_{11} = b_{13} = b_{21} = b_{23} = b_{31} = r_{23} = r''_{12} = 0$	a_2	T2(4)
(29)	8	$a_1 = c_{23} = b_{11} = b_{12} = b_{22} = b_{32} = d_{23} = g_{23} = 0$	b_{33}	T2(6)
(30)	8	$a_3 = c'''_{12} = b_{12} = b_{21} = b_{31} = b_{32} = b_{33} = g'''_{12} = 0$	$a_1 a_2$	Q
(31)	8	$a_2 = c''_{31} = b_{13} = b_{22} = b_{31} = d'_{21} = d'_{23} = g''_{31} = 0$	$a_1 a_3$	Q
(32)	8	$a_1 = b_{11} = b_{21} = b_{23} = b_{31} = d''_{12} = d_{13} = r'_{32} = 0$	$a_2 a_3 c''_{23}$	Q
(33)	8	$a_1 = c_{23} = b_{11} = b_{21} = b_{31} = d_{13} = j_{13} = n_{32} = 0$	$d''_{23} b_{22} b_{33}$	T4(2)
(34)	8	$a_2 = c'''_{13} = b_{13} = b_{21} = b_{22} = b_{23} = b_{31} = g'''_{13} = 0$	$a_1 a_3$	Q
(35)	8	$a_2 = c''_{13} = b_{11} = b_{21} = b_{22} = b_{31} = d_{13} = g''_{31} = 0$	$b_{23} g_{12}$	P1(7)
(36)	8	$a_3 = c''_{21} = b_{12} = b_{21} = b_{31} = b_{32} = b_{33} = g''_{21} = 0$	(*)	P1(7)
(37)	8	$a_1 = a_3 = b_{11} = b_{13} = b_{21} = b_{31} = b_{33} = d_{12} = 0$	$b_{22} b_{32}$	T2(6)
(38)	8	$a_1 = a_3 = b_{11} = b_{12} = b_{13} = b_{31} = b_{33} = d'_{32} = 0$	a_2	Q
(39)	8	$a_2 = a_3 = b_{12} = b_{22} = b_{23} = b_{32} = b_{33} = g_{23} = 0$	b_{21}	P1(8)
(40)	8	$a_2 = a_1 = b_{22} = b_{21} = b_{12} = b_{11} = d'_{23} = g_{12} = 0$	(**)	P1(10)
(41)	8	$a_1 = a_2 = b_{11} = b_{12} = b_{21} = b_{22} = b_{31} = d_{13} = 0$	$a_3 g_{12} b_{23} b_{13} b_{33}$	P1(8)
(42)	8	$a_1 = a_2 = a_3 = b_{33} = n_{31} = n_{23} = n'_{23} = o_{123} = 0$	$b_{12} b_{13} b_{23}$	T2(2)
(43)	8	$a_1 = a_2 = a_3 = b_{12} = b_{22} = b_{31} = b_{32} = b_{33} = 0$	b_{11}	T(2)
(44)	8	$a_1 = a_2 = a_3 = b_{11} = b_{21} = b_{22} = b_{23} = b_{31} = 0$	b_{33}	P1(4)
(45)	8	$a_1 = a_2 = a_3 = b_{21} = b_{33} = d''_{31} = n_{23} = o_{123} = 0$	b_{23}	P1(4)
(46)	8	$a_1 = a_2 = a_3 = b_{33} = d''_{32} = g''_{12} = n_{31} = o_{123} = 0$	$b_{11} b_{23}$	P1(4)

Table 4: Integrability conditions of Theorem 6 (continued).

st.	n	conditions =0	conditions $\neq 0$	f.i. type
(47)	8	$a_1 = a_2 = a_3 = b_{11} = b_{13} = b_{31} = b_{33} = d''_{32} = 0$	b_{22}	T2(2)
(48)	8	$a_1 = a_2 = a_3 = b_{11} = b_{13} = b_{31} = b_{32} = b_{33} = 0$	b_{12}	P1(4)
(49)	8	$a_1 = a_2 = a_3 = b_{21} = b_{22} = b_{23} = d_{31} = d_{13} = 0$	g_{31}	Q
(50)	8	$a_1 = a_2 = a_3 = b_{22} = b_{33} = e_{31} = g_{23} = n'_{23} = 0$	$b_{12}b_{23}b_{32}g'''_{31}$	Q
(51)	8	$a_1 = a_2 = a_3 = b_{11} = b_{33} = d''_{32} = g_{23} = n'_{23} = 0$	$b_{13}b_{23}b_{22}g''_{12}$	T4(1)
(52)	8	$a_1 = a_2 = a_3 = b_{13} = b_{23} = d''_{31} = d''_{32} = n'_{12} = 0$	$b_{11}b_{21}b_{33}$	Q
(53)	8	$a_1 = a_2 = a_3 = b_{22} = b_{33} = d''_{31} = g_{13} = n'_{31} = 0$	$b_{21}b_{23}d_{21}$	Q
(54)	9	$c_{12} = c'_{31} = b_{32} = b_{12} = b_{22} = d_{21} = d_{13} = d_{31} = d_{23} = 0$	$\lvert a_1\rvert + \lvert b_{11}\rvert + \lvert b_{33}\rvert$	T2(5)
(55)	9	$c_{12} = c_{13} = b_{11} = b_{21} = b_{31} = b_{32} = d''_{13} = d''_{12} = n_{23} = 0$	$b_{33}b_{22}$	Q
(56)	9	$c''_{13} = c''_{23} = b_{11} = b_{21} = b_{31} = d_{12} = d_{23} = n_{32} = d_{13} = 0$	$a_1a_2a_3$	Q
(57)	9	$a_1 = c''_{23} = b_{11} = b_{12} = b_{13} = b_{23} = b_{33} = e_{32} = g''_{23} = 0$		T2(1)
(58)	9	$a_3 = c''_{12} = b_{12} = b_{13} = b_{21} = b_{23} = b_{31} = b_{32} = b_{33} = 0$	$\lvert a_1\rvert + \lvert b_{11}\rvert + \lvert b_{22}\rvert$	T2(4)
(59)	9	$a_3 = c''_{21} = b_{23} = b_{33} = b_{13} = d_{32} = d_{31} = d_{12} = d_{21} = 0$	a_1a_2	Q
(60)	9	$a_1 = c'''_{23} = b_{11} = b_{21} = b_{23} = b_{31} = b_{32} = d_{12} = d''_{13} = 0$	a_2a_3	Q
(61)	9	$a_2 = a_1 = b_{22} = b_{21} = b_{32} = b_{12} = b_{13} = b_{11} = d_{23} = 0$	a_3	P1(8)
(62)	9	$a_1 = a_3 = b_{11} = b_{13} = b_{23} = b_{31} = b_{33} = d''_{12} = d_{32} = 0$	a_2	Q
(63)	9	$a_1 = a_2 = a_3 = b_{11} = b_{22} = b_{33} = b_{23} = b_{32} = g_{23} = 0$	$\lvert b_{21}\rvert + \lvert b_{12}\rvert + \lvert b_{13}\rvert$	T2(2)

(*) $|a_1| + |b_{11}| + |b_{22}| + |b_{23}| \neq 0$ (**) $|a_3| + |b_{31}| + |b_{32}| + |b_{33}| \neq 0$

Theorem 7. *An LV3 is integrable with two first integrals one of which being of the type of Theorem 2(2) in the cases of Table 5, which exclude those already mentioned in Theorem 6.*

Table 5. Integrability conditions of Theorem 7.

st.	n	conditions =0	conditions $\neq 0$	f.i. type
(1)	5	$c_{12} = c_{13} = d_{12} = j_{12} = o_{123} = 0$		Q
(2)	6	$c_{12} = c_{23} = b_{12} = b_{32} = r_{31} = o_{123} = 0$	$b_{11}b_{31}b_{33}d_{23}$	Q
(3)	6	$c_{12} = c_{13} = o_{123} = d'_{31} = d'_{21} = r_{32} = 0$	$b_{11}b_{33}$	T4(49)
(4)	6	$c_{12} = c_{13} = b_{33} = n'_{12} = n'_{23} = o_{123} = 0$	d_{31}	T2(4)
(5)	6	$a_1 = a_2 = a_3 = g_{23} = h_{23} = o_{123} = 0$	$b_{22}b_{33}d_{13}d_{32}$	T2(3)
(6)	7	$c_{12} = c_{13} = b_{13} = b_{23} = b_{31} = b_{32} = r_{12} = 0$	$a_1a_2a_3b_{22}b_{12}b_{21}$	Q
(7)	7	$a_1 = a_2 = a_3 = b_{13} = b_{23} = j_{23} = g_{31} = 0$	(*)	T2(3)
(8)	7	$a_1 = a_2 = a_3 = b_{12} = b_{13} = b_{23} = o_{123} = 0$	$b_{33}d_{21}j_{23}$	T2(3)
(9)	7	$a_1 = a_2 = a_3 = b_{22} = b_{33} = g_{23} = o_{123} = 0$	$b_{11}b_{23}$	T2(3)
(10)	8	$c_{12} = c_{13} = b_{11} = b_{22} = b_{31} = d''_{23} = g'_{21} = g''_{31} = 0$	$b_{21}b_{33}$	T2(4)
(11)	8	$a_1 = a_2 = a_3 = b_{12} = d_{13} = d_{31} = d'_{32} = p'''_{13} = 0$	$b_{11}d_{23}$	T4(4)
(12)	8	$a_1 = a_2 = a_3 = b_{13} = b_{22} = b_{23} = i_{32} = o_{123} = 0$	$b_{21}d_{21}$	T2(3)
(13)	8	$a_1 = a_2 = a_3 = b_{11} = b_{32} = b_{33} = g_{32} = o_{123} = 0$	b_{22}	T2(3)
(14)	8	$a_1 = a_2 = a_3 = b_{21} = d_{12} = d_{23} = e_{31} = g_{13} = 0$	b_{33}	T2(3)
(15)	8	$a_1 = a_2 = a_3 = b_{32} = b_{33} = d'_{21} = g_{23} = o_{123} = 0$	$b_{11}b_{22}d_{12}$	T2(3)
(16)	8	$a_1 = a_2 = a_3 = b_{22} = b_{23} = d'_{21} = d'_{31} = o_{123} = 0$	b_{33}	T2(3)
(17)	8	$a_1 = a_2 = a_3 = b_{11} = b_{33} = d''_{32} = g_{23} = o_{123} = 0$	$b_{13}b_{23}b_{22}$	T2(3)
(18)	8	$a_1 = a_2 = a_3 = b_{22} = b_{33} = d''_{31} = g_{13} = o_{123} = 0$	$b_{11}d_{21}$	T2(3)
(19)	9	$a_1 = a_2 = a_3 = b_{11} = b_{12} = b_{33} = d''_{32} = g_{12} = g_{23} = 0$		T2(3)

(*) $d_{31}d_{32}(b_{11}b_{22} + b_{32}d''_{31}) \neq 0$

Theorem 8. *An LV3 is integrable with two first integrals one of which being of the type of Theorem 2(3)(excluding those included in Theorem 7) in the cases of Table 6.*

Table 6: Integrability conditions of Theorem 8.

st.	n	conditions =0	conditions $\neq 0$	f.i. type
(1)	5	$a_1 = a_2 = a_3 = g_{12} = i'''_{23} = 0$	$d_{23}j_{13}$	Q
(2)	6	$a_1 = c_{23} = b_{11} = b_{32} = e_{12} = g_{23} = 0$	$b_{33}d_{23}d_{32}d_{13}j_{12}$	Q
(3)	6	$a_1 = c_{23} = b_{11} = b_{21} = b_{31} = b_{32} = 0$	(*)	P1(2),T2(3)
(4)	6	$a_1 = c_{23} = b_{11} = g_{13} = d'_{12} = g_{23} = 0$	$a_2a_3d_{23}d_{32}d_{13}d''_{23}$	Q
(5)	6	$a_1 = c_{23} = b_{11} = b_{32} = o_{123} = g_{23} = 0$	$d_{23}d_{32}d_{13}$	Q
(6)	6	$a_1 = c_{23} = b_{11} = b_{12} = d'_{32} = g_{23} = 0$	$d_{13}d''_{23}j_{12}$	Q
(7)	6	$a_1 = c_{23} = b_{11} = b_{13} = d'_{23} = g_{23} = 0$	$b_{12}d_{12}d''_{32}j_{13}$	Q
(8)	6	$a_1 = c_{23} = b_{11} = g_{23} = n_{23} = o_{123} = 0$	$b_{22}b_{33}d_{23}d_{32}d_{12}d''_{32}$	Q
(9)	6	$a_1 = c_{23} = b_{11} = b_{12} = b_{32} = g_{23} = 0$	$b_{22}b_{33}b_{13}d_{13}$	T2(7)
(10)	6	$a_2 = c_{13} = b_{22} = g_{23} = g_{13} = r_{31} = 0$	b_{33}	Q
(11)	6	$a_1 = c_{23} = b_{11} = n'_{23} = r_{23} = g_{23} = 0$	$b_{22}b_{33}d_{13}d_{32}h'_{23}$	Q
(12)	6	$a_1 = c_{23} = b_{11} = b_{13} = b_{23} = g_{23} = 0$	$b_{12}d_{32}$	P1(8)
(13)	6	$a_1 = c_{23} = b_{11} = g_{23} = g_{13} = r_{23} = 0$	$a_2a_3d_{13}d_{12}$	Q
(14)	6	$a_1 = a_2 = a_3 = g_{12} = g_{23} = g_{31} = 0$	$b_{21}d_{23}d_{32}$	Q
(15)	6	$a_1 = a_2 = a_3 = b_{13} = b_{23} = g_{23} = 0$	(**)	T2(3)
(16)	7	$a_1 = c_{23} = b_{11} = b_{32} = b_{12} = g_{12} = g_{23} = 0$	$b_{33}b_{21}d_{23}d_{32}$	Q
(17)	7	$a_1 = c'''_{12} = c''_{13} = h'_{31} = p_{32} = g_{32} = g_{31} = 0$	$b_{11}b_{22}b_{33}d_{13}d_{12}$	T2(3)
(18)	7	$a_1 = a_2 = a_3 = d'_{32} = d'_{12} = n_{31} = q'''_{31}$	(***)	Q
(19)	7	$a_1 = a_2 = a_3 = g_{23} = n_{23} = n'_{23} = p^{iv}_{13} = 0$	$b_{12}b_{13}b_{22}d_{23}$	T4(1)
(20)	7	$a_1 = a_2 = a_3 = b_{12} = b_{13} = n_{23} = g_{32} = 0$	$b_{22}b_{33}d_{32}d''_{32}$	T4(1)
(21)	7	$a_1 = a_2 = a_3 = b_{12} = b_{32} = b_{33} = g_{23} = 0$	$b_{21}b_{13}$	T2(3)
(22)	7	$a_1 = a_2 = a_3 = d'_{21} = d'_{31} = n_{23} = q'''_{23} = 0$	$b_{33}d_{32}d''_{32}i''_{31}$	Q
(23)	7	$a_1 = a_2 = a_3 = b_{13} = b_{22} = d'_{23} = g_{23} = 0$	(****)	Q
(24)	7	$a_1 = a_2 = a_3 = b_{33} = g_{31} = g_{32} = q'_{21} = 0$	$b_{13}b_{23}d_{12}g''_{12}$	Q
(25)	8	$c_{23} = c'_{31} = b_{11} = b_{21} = b_{31} = b_{32} = d'_{12} = d_{13} = 0$	$a_1a_2a_3b_{33}d_{23} \neq 0$	P1(2),T2(3)
(26)	8	$c_{12} = c_{23} = b_{11} = r_{21} = r_{31} = r'_{23} = r'_{32} = n'_{13} = 0$	$b_{22}b_{33}$	T4(37)
(27)	8	$a_1 = c_{23} = b_{11} = b_{13} = b_{21} = b_{31} = d'_{23} = j_{13} = 0$	a_2a_3	Q
(28)	8	$a_1 = a_2 = a_3 = b_{33} = e_{31} = e_{32} = g_{13} = g_{23} = 0$	$b_{23}b_{31}b_{21}d_{23}$	T2(3)
(29)	8	$a_1 = a_2 = a_3 = b_{13} = b_{22} = b_{23} = d''_{31} = g_{13} = 0$	b_{21}	T2(3)
(30)	8	$a_1 = a_2 = a_3 = b_{22} = d''_{21} = d'_{13} = g_{12} = g_{23} = 0$	g'_{31}	T2(3)
(31)	8	$a_1 = a_2 = a_3 = b_{11} = b_{22} = b_{33} = g_{23} = g_{13} = 0$	g''_{12}	T2(3)
(32)	8	$a_1 = a_2 = a_3 = b_{11} = b_{22} = b_{33} = g_{23} = n'_{23} = 0$	$b_{23}b_{21}g''_{12}$	Q
(33)	9	$c''_{12} = c''_{23} = b_{12} = b_{22} = b_{32} = d_{23} = e_{31} = g_{23} = r_{31} = 0$	b_{33}	Q
(34)	9	$a_1 = c_{23} = c'''_{31} = b_{11} = b_{31} = d''_{12} = d_{32} = d_{23} = d_{13} = 0$	$b_{21}b_{33}$	Q
(35)	9	$a_1 = a_2 = a_3 = b_{33} = d'_{12} = d''_{21} = d''_{32} = g_{32} = g'''_{21} = 0$	b_{11}	Q
(36)	9	$a_1 = a_2 = a_3 = d_{12} = d_{13} = d_{21} = d_{31} = n_{32} = j'_{12} = 0$	$b_{33}d_{13}$	Q
(37)	9	$a_1 = a_2 = a_3 = b_{22} = b_{23} = d''_{21} = d'_{13} = g_{23} = g'''_{31} = 0$	b_{12}	Q
(38)	9	$a_1 = a_2 = a_3 = b_{22} = b_{33} = d''_{31} = g_{23} = g'''_{21} = g'''_{31} = 0$	$b_{12}b_{23}$	Q

(*) $b_{22}b_{33}d_{23}(b_{22}b_{13} + b_{33}b_{12} - b_{23}b_{12}) \neq 0$ (**) $d_{12}d_{21}d_{32}(b_{12}b_{21} - b_{11}b_{22}) \neq 0$
(***) $b_{11}d_{13}d''_{13}i''_{12}(2d_{13} - b_{33}) \neq 0$ (****) $b_{12}g''_{31}(b_{12}b_{21} + b_{11}b_{32}) \neq 0$.

Theorem 9. *An LV3 is integrable with two first integrals one of which being of the type of Theorem 2(4) in the cases of Table 7.*

Table 7: Integrability conditions of Theorem 9.

st.	n	conditions =0	conditions $\neq 0$	f.i. type
(1)	6	$b_{11} = b_{12} = b_{21} = b_{31} = b_{32} = r_{23} = 0$	$a_2 b_{22} b_{33}$	T2(7)
(2)	6	$b_{13} = b_{21} = b_{23} = b_{31} = r_{12} = r_{23} = 0$	$a_1 a_2 a_3$	T2(4)
(3)	6	$b_{13} = b_{21} = b_{23} = e_{31} = r_{12} = r_{32} = 0$	$a_2 b_{11}$	Q
(4)	6	$c_{12} = c'_{32} = b_{13} = b_{23} = o_{123} = r_{12} = 0$	$a_1 a_2 a_3 b_{11} d_{21}$	Q
(5)	6	$a_1 = c_{23} = b_{11} = g_{23} = g_{13} = r_{23} = 0$	$b_{22} d_{12}$	P1(3)
(6)	7	$c'''_{23} = b_{12} = b_{31} = b_{32} = e_{21} = e_{23} = r_{31} = 0$	$a_2 a_3 c''_{12}$	Q
(7)	7	$c''_{31} = b_{12} = b_{13} = b_{31} = b_{32} = e_{21} = r_{23} = 0$	$a_1 a_2 a_3 c''_{12}$	Q
(8)	7	$c''_{23} = b_{23} = b_{31} = b_{32} = b_{12} = e_{21} = r_{13} = 0$	$a_1 a_2 a_3 c''_{12}$	Q
(9)	8	$c''_{12} = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d_{13} = r_{32} = 0$	$a_1 a_2$	P1(2)
(10)	8	$c''_{31} = b_{11} = b_{21} = b_{31} = b_{32} = d_{12} = d_{13} = r_{23} = 0$	$a_1 a_3 b_{33}$	P1(2)
(11)	8	$c''_{13} = b_{11} = b_{13} = b_{21} = b_{23} = b_{31} = r_{23} = r''_{12} = 0$	$a_1 a_2 a_3$	P1(2)
(12)	8	$c_{23} = b_{12} = b_{22} = b_{31} = b_{32} = d_{23} = e_{21} = s''_{13} = 0$	b_{23}	P1(6)
(13)	8	$c''_{12} = c_{13} = b_{12} = b_{21} = b_{23} = b_{32} = e_{31} = r'_{13} = 0$	$a_1 a_2 a_3$	T5(19)
(14)	8	$c'''_{12} = c_{32} = b_{12} = b_{13} = b_{32} = d'_{21} = d''_{23} = d'_{31} = 0$	$a_1 a_2 a_3$	T5(14)
(15)	8	$c_{23} = c'''_{13} = b_{12} = b_{13} = b_{23} = d'_{21} = d''_{32} = g_{32} = 0$	$a_1 a_2 a_3$	Q
(16)	8	$c_{12} = c'''_{23} = b_{12} = b_{13} = b_{21} = b_{23} = b_{32} = d'_{31} = 0$	$a_1 a_2 a_3$	T5(43)
(17)	8	$a_1 = b_{11} = b_{21} = b_{32} = b_{31} = d_{13} = d'_{12} = r_{23} = 0$	$\|a_3\| + \|b_{33}\|$	P1(2)
(18)	8	$a_3 = b_{13} = b_{21} = b_{23} = b_{33} = d_{32} = e_{31} = r_{12} = 0$	a_2	P1(2)
(19)	8	$a_3 = b_{33} = b_{13} = b_{23} = b_{21} = d_{31} = d''_{32} = r_{12} = 0$	$a_1 a_2$	P1(2)
(20)	8	$a_1 = c_{23} = b_{11} = b_{21} = b_{31} = d_{13} = j_{13} = r_{32} = 0$	b_{22}	P1(3)
(21)	8	$a_2 = b_{12} = b_{21} = b_{22} = b_{31} = b_{32} = d_{23} = r_{31} = 0$	$a_1 a_3$	P1(2)
(22)	8	$a_1 = b_{11} = b_{21} = b_{31} = b_{32} = d_{13} = d''_{12} = s_{23} = 0$	$a_2 a_3$	P1(2)
(23)	9	$a_3 = c'''_{12} = b_{12} = b_{13} = b_{23} = b_{31} = b_{33} = d'_{21} = d_{32} = 0$	$\|a_1\| + \|b_{11}\| + \|b_{22}\|$	Q
(24)	9	$c_{23} = c'_{31} = b_{12} = b_{13} = b_{23} = b_{32} = b_{33} = d_{31} = r_{12} = 0$	$b_{11} b_{22}$	T2(5)
(25)	9	$c''_{12} = c_{13} = b_{11} = b_{21} = b_{23} = b_{31} = b_{32} = d_{12} = d_{13} = 0$	$a_1 a_2 a_3$	P1(2)
(26)	9	$c_{12} = c_{23} = b_{11} = b_{21} = b_{31} = b_{32} = d_{12} = d_{13} = d''_{23} = 0$	$a_1 a_2 a_3 b_{22} b_{33}$	P1(2,3)
(27)	9	$c_{13} = c_{23} = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d''_{13} = d''_{32} = 0$	(*)	P1(2,3)
(28)	9	$c_{23} = c'_{21} = b_{12} = b_{22} = b_{31} = b_{32} = d_{21} = d_{23} = j''_{12} = 0$	$\|a_1\| + \|b_{11}\| + \|b_{33}\|$	T4(4)
(29)	9	$c_{12} = c_{13} = b_{12} = b_{21} = b_{22} = b_{32} = d_{23} = d'_{31} = r_{13} = 0$	b_{11}	Q
(30)	9	$c'_{21} = c_{23} = b_{12} = b_{31} = b_{22} = b_{32} = d_{23} = d''_{21} = r_{31} = 0$	b_{13}	Q
(31)	9	$a_3 = c''_{21} = b_{23} = b_{21} = b_{33} = b_{12} = b_{13} = d_{32} = d''_{31} = 0$	$\|a_1\| + \|b_{11}\|$	P1(2)
(32)	9	$a_1 = c''_{23} = b_{11} = b_{13} = b_{21} = b_{23} = b_{31} = b_{32} = d''_{12} = 0$	$\|a_2\| + \|b_{33}\|$	P1(2)
(33)	9	$a_1 = c_{23} = b_{11} = b_{21} = b_{31} = d_{13} = d'_{12} = d'_{23} = n'_{32} = 0$	$b_{22} b_{33}$	P1(3)
(34)	9	$a_1 = c''_{23} = b_{11} = b_{21} = b_{23} = b_{31} = b_{32} = d_{12} = d''_{13} = 0$	$\|a_2\| + \|b_{33}\|$	P1(2)
(35)	9	$a_3 = c''_{12} = b_{12} = b_{13} = b_{21} = b_{23} = b_{33} = d_{32} = e_{31} = 0$	$\|a_2\| + \|b_{22}\|$	P1(2)
(36)	9	$a_2 = c'''_{13} = b_{12} = b_{22} = b_{31} = b_{32} = d_{21} = d''_{23} = e_{13} = 0$	$\|a_3\| + \|b_{33}\|$	P1(2)
(37)	9	$a_1 = c''_{23} = b_{11} = b_{21} = b_{23} = b_{31} = b_{32} = d_{12} = d''_{13} = 0$	$\|a_3\| + \|b_{33}\|$	P1(2)
(38)	9	$a_2 = c_{13} = b_{12} = b_{22} = b_{32} = d_{23} = e_{31} = g_{23} = r_{13} = 0$	$\|a_1\| + \|b_{11}\| + \|b_{33}\|$	Q
(39)	9	$a_2 = c'''_{13} = b_{12} = b_{13} = b_{21} = b_{22} = b_{32} = d'_{31} = d_{23} = 0$	$\|a_1\| + \|b_{11}\| + \|b_{33}\|$	Q
(40)	9	$a_3 = c'''_{21} = b_{12} = b_{13} = b_{23} = b_{32} = b_{33} = e_{21} = e_{31} = 0$	$\|a_1\| + \|b_{11}\| + \|b_{22}\|$	Q
(41)	9	$a_3 = c'''_{12} = b_{12} = b_{13} = b_{23} = b_{31} = b_{33} = d'_{21} = d''_{32} = 0$	$a_1 a_2$	Q

(*) $(a_2 + b_{22})(b_{22} + b_{33}) \neq 0$

Theorem 10. *The LV3 has two first integrals with at least one being formed by one linear algebraic solution not included in Theorems 6-9 in the cases of Table 8.*

Table 8. Integrability conditions of Theorem 10.

st.	n	conditions =0	cond. $\neq 0$	f.i. type
(1)	5	$a_1 = b_{11} = b_{12} = b_{32} = d''_{13}$	$a_3 b_{31} b_{33}$	T2(7),Q
(2)	6	$b_{11} = b_{12} = b_{21} = b_{31} = b_{32} = r'_{23} = 0$	$a_2 a_3 c_{23}$	T2(7),T5(1)
(3)	6	$a_1 = c'_{32} = b_{11} = b_{13} = b_{23} = g'_{32} = 0$	$b_{12} b_{22} d_{12} d''_{32}$	2T2(7)
(4)	7	$c''_{23} = b_{23} = b_{31} = b_{32} = b_{13} = d_{21} = d_{12} = 0$	$a_1 a_2 a_3 c_{12}$	T2(5),Q
(5)	8	$c_{23} = b_{12} = b_{13} = b_{22} = b_{31} = b_{32} = e_{21} = d''_{23} = 0$	c_{12}	2P1(2)
(6)	8	$c'_{31} = b_{12} = b_{13} = b_{22} = b_{23} = b_{32} = b_{33} = d_{31} = 0$	l_{32}	T2(5),Q
(7)	8	$c'_{21} = c_{23} = b_{12} = b_{13} = b_{22} = b_{32} = d_{23} = d_{21} = 0$	$b_{21} b_{23}$	T2(11),Q
(8)	8	$c_{12} = c_{23} = b_{12} = b_{22} = b_{31} = b_{32} = d_{21} = d_{23} = 0$	$a_1 a_2 a_3 b_{11}$	T2(11),Q
(9)	8	$c_{32} = c''_{13} = b_{23} = b_{31} = b_{32} = b_{13} = d_{21} = d_{12} = 0$	b_{11}	T2(5),T5(3)
(10)	8	$a_1 = b_{31} = b_{32} = b_{11} = b_{12} = b_{23} = b_{21} = d_{13} = 0$	$a_2 a_3$	T2(7),Q
(11)	8	$a_1 = c_{23} = b_{11} = b_{13} = b_{21} = b_{31} = d_{12} = d'_{23} = 0$	b_{22}	P1(3),Q
(12)	8	$a_1 = c'''_{32} = b_{11} = b_{12} = b_{32} = d'_{13} = g_{12} = g'''_{32} = 0$	$a_2 a_3$	T2(6),Q
(13)	8	$a_1 = c'''_{32} = b_{11} = b_{12} = b_{23} = b_{32} = e_{13} = g'''_{32} = 0$	$a_2 a_3$	T2(6),Q
(14)	8	$a_1 = c'''_{32} = b_{11} = b_{12} = b_{23} = b_{32} = d'_{13} = g'''_{32} = 0$	$a_2 a_3$	T2(6),Q
(15)	9	$c_{12} = c_{13} = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d''_{13} = n_{23} = 0$	$a_1 a_2 a_3$	P1(2),Q
(16)	9	$a_1 = c'''_{32} = b_{11} = b_{13} = b_{21} = b_{23} = b_{31} = b_{32} = d''_{12} = 0$	$\lvert a_3 \rvert + \lvert b_{33} \rvert$	P1(2),Q
(17)	9	$a_1 = c''_{23} = b_{11} = b_{21} = b_{31} = d''_{12} = d''_{13} = d_{23} = d_{32} = 0$	$a_2 a_3$	T2(5),Q

Theorem 11. *The LV3 has two first integrals with quadratic algebraic solutions in the cases of Table 9.*

Table 9. Integrability conditions of Theorem 11.

st.	n	conditions =0	cond. $\neq 0$	f.i. type
(1)	7	$c_{23} = b_{12} = b_{13} = d'_{23} = d'_{32} = g_{23} = r_{13} = 0$	(*)	Q,Q
(2)	7	$c'''_{23} = b_{31} = b_{32} = d_{12} = d_{21} = e_{23} = n'_{13} = 0$	$a_1 a_2 a_3 b_{11} b_{33}$	Q,Q
(3)	7	$c_{23} = c'_{21} = d_{21} = d_{12} = d_{31} = r_{23} = r_{13} = 0$	$b_{11} b_{22} b_{33} d_{23}$	T4(12),Q
(4)	7	$c_{23} = c'_{21} = b_{12} = g_{32} = n_{12} = r_{23} = n_{31} = 0$	$d_{32} d''_{31}$	T4(12),T4(28)
(5)	7	$c''_{12} = c'_{13} = b_{23} = b_{31} = e_{12} = e_{21} = e_{32} = 0$	$b_{13} b_{33} d''_{13}$	T5(30),Q
(6)	8	$c_{12} = c_{23} = b_{13} = r_{21} = r_{31} = r'_{23} = r'_{32} = n'_{13} = 0$	$b_{22} b_{33}$	T4(37),T4(1)
(7)	8	$c''_{12} = c''_{13} = b_{31} = b_{21} = r_{12} = r_{13} = r'_{23} = r'_{32} = 0$	$a_1 a_2 a_3 b_{11} b_{12}$	T4(2),T5(46)
(8)	8	$c_{12} = c_{13} = b_{21} = b_{23} = b_{31} = g_{13} = n_{12} = n'_{23} = 0$	$b_{22} b_{33}$	T4(1), T4(2)
(9)	8	$a_3 = c'''_{12} = b_{12} = b_{21} = b_{32} = b_{33} = e_{31} = g'''_{12} = 0$	$a_1 a_2$	Q,Q
(10)	8	$a_2 = c'''_{31} = b_{21} = b_{22} = b_{31} = d'_{13} = d''_{23} = n'_{23} = 0$	$a_1 a_3$	T5(4),Q
(11)	8	$a_3 = c''_{12} = b_{32} = b_{33} = d_{21} = d_{12} = g'_{32} = g''_{12} = 0$	a_1	T5(56),T5(57)
(12)	9	$c''_{12} = c'_{13} = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d''_{13} = r'_{32} = 0$	$a_1 a_2 a_3$	Q,Q
(13)	9	$c'_{21} = c'''_{23} = b_{11} = b_{21} = b_{23} = b_{31} = d_{13} = d'_{12} = n_{23} = 0$	$a_1 a_2 a_3$	T5(5), Q
(14)	9	$c'_{21} = c'_{31} = b_{12} = b_{13} = b_{22} = b_{32} = d''_{21} = d''_{23} = n_{31} = 0$	$a_1 a_2 a_3 b_{33}$	T4(4),T4(27)
(15)	9	$c_{13} = c_{23} = b_{11} = b_{21} = b_{23} = b_{31} = d_{12} = d''_{13} = n_{23} = 0$	$b_{22} b_{33}$	Q,Q
(16)	9	$c''_{12} = c_{23} = b_{11} = b_{21} = b_{31} = d_{12} = d_{13} = r'_{23} = n_{23} = 0$	$a_1 a_2 a_3$	T5(10),T5(46)

(*) $a_1 b_{11} d_{21} d_{31} d''_{23} d''_{32} \neq 0$.

We note, before concluding, that Theorem 4(6) and Theorem 8(1) with the additional conditions $b_{11} = b_{22} = b_{33} = 0$ and Theorem 8(31–32) are Theorem 6(5–6) and Theorem 7(4) and 7(7) respectively of [4] in the case of ABC systems. Moreover, we note that to

the above mentioned cases of Theorem 11, one must add the case obtained by Moulin–Ollagnier [30] for the ABC system which is formed with the first integral of Theorem 4(5) and an other one which is not of Darboux type (see Ref. 31 for more details).

7 Conclusion

The use of Darboux method for the LV3 has been rather efficient to find first integrals and to detect integrable cases. The main reason of this efficiency must be found in the fact that the problem of searching a first integral has been reduced here to the search of only one algebraic solution. (Theorem 3 is when a quadratic algebraic solution can be factorised in two linear ones). In total we have found 366 cases, among which 172 are cases of a single first integral and 194 when two first integrals coexist. As predicted, we see that for the existence of quadratic invariant solutions and the corresponding first integrals, we require a larger number of conditions than for linear algebraic solutions. In fact for having a single first integral formed by planes we can need only a maximum of 3 conditions in total (Theorem 2 (1)-(4)), whereas it is required 4 to 8 conditions when at least there is one second degree solution. The cases of integrability require at least 4 conditions when both first integrals are linear (in fact one case with 4 conditions and 5 with 5 conditions). Otherwise their number can grow now to 9 conditions. In general the number of conditions required is the number corresponding to the more constrained first integral increased by one. The number of essential parameters of the LV3 system being 9 (8 if we normalize the time), we omitted to consider the cases having more than 9 conditions. The present work can be completed with the cases where the planes $x = 0$, $y = 0$, $z = 0$ and $f_4 = 0$ are replaced by other planes and polynomials of greater degree, by exponential factors (see Cairó and Llibre [4]) and with time-dependent first integrals.

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